More Algebras for Determiners

R. Zuber

CNRS, Paris
Richard.Zuber@linguist.jussieu.fr

Abstract. Some new algebras, which are possible denotations for various determiners, are studied. One of them is the algebra of generalised cardinal quantifiers which is a sub-algebra of conservative quantifiers and which contains cardinal, co-cardinal and proportional quantifiers. In addition some non-conservative quantifiers are studied (symmetric, contrapositional and fixed points with respect to the post-complement). It is shown that co-intersective quantifiers are contrapositional. The analysis is extended to quantifiers of higher types.

1 Introduction

The study of quantification in natural language (NL) within the framework of generalised quantifier theory (GQT) is more than a quarter of century old. Inspired by the results and methods of Mostowski ([13]) it has given rise to many unforeseen secure formal and empirical results against a background of the classical quantificational logic. The range of problems treated in GQT is very large. It concerns not only traditional logical questions such as expressive power, first order definability or variable binding but also the documentation, classification and study of rich variety of quantifiers in different NLs ([1] [5], [6], [7], [8],[10], [11],[12], [15], [16],[18]). This classificatory and explanatory work shows that NL quantifiers exhibit many specific properties and it has been conjectured that some of these properties may characterise quantifiers in all NLs. For instance the property of *conservativity* (see below) has been considered a semantic universal in the sense that all quantifiers in all NLs should satisfy it. More recent empirical research ([7], [18]) shows that conservativity should not be considered as an "absolute" universal but a kind of relative universal. This roughly means that violations of conservativity are not arbitrary but are due to very specific contexts which still impose constraints on quantifiers which are possibly weaker or of a different nature. Unfortunately I will not say much more about this problem.

Another general result in GQT is that many of the constraints are Boolean in nature: this means that the sets of objects satisfying a particular constraint form Boolean algebras. This fact is of course compatible with a general observation that NL grammatical categories have Boolean structure ([9]). What is interesting, however, in this case is the fact that various constraints on quantifiers are often logically related, thus giving rise to an array of sub-algebras

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of a fixed Boolean algebra. This means in particular that there are expressions denoting quantifiers belonging to many algebras at the same time. Moreover, as we will see, there are expressions which denote quantifiers which are atoms in one algebra and which are not atoms in the corresponding super-algebra. A look at the possible status of such expressions and their relevance in linguistics does not seem pointless.

The purpose of this paper is to prepare formal tools which can help to cope with these two problems: (1) how the constraints of conservativity, or similar "universalistic" constraints can be relativised or superseded, and (2) what is the linguistic status of expressions denoting in different but related denotational algebras. Although I will use many linguistic examples I am not concerned here with applicational issues but basically by presentation of various algebraic properties of quantifiers considered as possible denotations of NL determiners.

Most results presented in this paper are not technically complex and will be given without proofs as obvious facts. Although many results obtained are also valid for infinite universes I will implicitly assume their finiteness. There are various reasons for this. First, although I am not directly concerned with issues of computational complexity, obviously one can naturally approach such issues only in finite contexts. Second, the finite-infinite distinction surely involves a specific competence which very likely is not a linguistic competence. Finally, there is a large and important class quantifiers, *proportional* quantifiers which can be defined only in finite domains. I discuss this class at some length.

2 Formal Preliminaries

The version of formal semantics adopted here is the one developed by Keenan (cf. [9]). Expressions of category C have their denotational algebra D_C, the set of possible denotations. D_C are atomic (and complete) Boolean algebras. Functional categories denote in $D_{B/A}$. The set of functions from A into B will be noted as $[A \rightarrow B]$. Atomicity of $D_{B/A}$ is inherited from the atomicity of D_B. When there are no constraints on functions from D_A on D_B, atoms of $D_{B/A}$ are determined by atoms of D_B in the following way:

Proposition: For any $a \in D_A$, and for any $\alpha \in AT(D_B)$, the function $f_{a,\alpha} \in D_{B/A}$ defined as $f_{a,\alpha}(x) = \alpha$ if $x = a$, and O_{D_B} otherwise, is an atom of $D_{B/A}$. Furthermore, every element of $D_{B/A}$ contains an atom of this form.

Noun phrases (NPs) denote functions from properties onto truth values. They are called quantifiers of type $\langle 1 \rangle$. They are elements of D_{NP} which are sets of sets. According to the above result, for any property P, the function f_P defined as $f_P(X) = 1$ if $X = P$ and $f_P(X) = O$ if $X \neq P$ is an atom of D_{NP}. Thus atoms of D_{NP} are singletons which contain a set as their unique element.

In GQT quantifiers are, roughly speaking, denotations of NPs or of their syntactic parts. Formally they are relations between sets. They can be expressed as functions into truth-values. For instance quantifiers of type $\langle 1 \rangle$ are functions from $P(E)$, the power set of E, the universe of objects, into $\{0, 1\}$.

In what follows we will be basically interested in the logic of denotations of (unary) nominal determiners. These are expressions (like *every, no*) which combine with common nouns to form noun phrases. Thus, semantically, they are functions from $P(E)$ onto type $\langle 1 \rangle$ quantifiers. They are called type $\langle 1, 1 \rangle$ quantifiers. These quantifiers can be viewed as binary relations on sets. Indeed a type $\langle 1, 1 \rangle$ quantifier F, which is a function in $[P(E) \rightarrow [P(E) \rightarrow \{0, 1\}]]$ corresponds to the binary relation Q between sets defined by $QXY \Leftrightarrow F(X)(Y) = 1$. Let us denote the set of all type $\langle 1, 1 \rangle$ quantifiers, or functions from $[P(E) \rightarrow [P(E) \rightarrow \{0, 1\}]]$ by $PDET$. This set forms, given Proposition, an atomic Boolean algebra with Boolean operations defined pointwise.

Notice that Boolean character of quantifiers and their arguments allows us to distinguish two types of negations or complements. The first is the usual Boolean complement. The second, called *postcomplement* is defined by the complement of one of its arguments. More precisely, let F be a type $\langle 1, 1 \rangle$ quantifier. Then its postcomplement, noted $F - not$, is the type $\langle 1, 1 \rangle$ quantifier defined as $F - not(X)(Y) = F(X)(Y')$. Similarly, if Q is a type $\langle 1 \rangle$ quantifier, then its postcomplement $Q - not$ is defined as $Q - not(Y) = Q(Y')$, or, equivalently: $Y \in Q - not$ iff $Y' \in Q$. Roughly speaking, syntactically, the postcomplement corresponds to the negation of verb phrases. In what follows we will use in particular the fact that postcomplements of meets or joins of two functions are respectively meets and joins of postcomplements of these functions.

One of the best known constraints on possible denotations of determiners is the constraint of conservativity. Since I am going to use two notions of conservativity, which depend on which specific intersection of arguments is taken into consideration, I will refer to this "classical" notion of conservativity by $CONS1$. By definition:

D1: $F \in CONS1$ iff for any property X, Y and Z if $X \cap Y = X \cap Z$ then $F(X)(Y) = F(X)(Z)$

There are two other equivalent definitions of conservativity indicated by the following facts:

Fact 1: (cf. [9]) $F \in CONS1$ iff for any property X, Y, $F(X)(Y) = F(X)(X \cap Y)$

Fact 2: $F \in CONS1$ iff for any property X, Y one has $F(X)(Y) = F(X)(X' \cup Y)$

The algebra $CONS1$ has two sub-algebras, the algebra INT of intersective functions, and the algebra $CO - INT$ of co-intersective functions ([5]):

D2: $F \in INT$, iff for all properties X, Y, Z and W, if $X \cap Y = Z \cap W$ then $F(X)(Y)$ is true iff $F(Z)(W)$ is true.

D3: $F \in CO - INT$ iff for all properties X, Y, Z and W, if $X - Y = Z - W$ then $F(X)(Y) = F(Z)(W)$.

Both sets INT and $CO-INT$ form atomic (and complete) Boolean algebras. Their atoms are determined by a property: for any property P the function F_P such that $F_P(X)(Y) = 1$ iff $X \cap Y = P$ is a atom of INT and the function F_P such that $F_P(X)(Y) = 1$ iff $X - Y = P$ is an atom of $CO - INT$. Exclusion determiners (cf.[16]) denote such atomic functions: *no...except Leo and Lea* denotes an atomic intersective function determined by the set composed of two elements, Leo and Lea.

Algebras INT and $CO-INT$ are important for two reasons. First, as shown in [5] the algebra $CONS1$ is a Boolean closure of INT and $CO-INT$. Second, Keenan also shows that quantifiers belonging to INT or to $CO-INT$ are exactly those which are sortally reducible. This means that it is possible to replace *salva veritate* the first argument of these quantifiers by the universal property E and replace the second argument by a Boolean combination of the first argument with the second: if $F \in INT$ then $F(X)(Y) = F(E)(X \cap Y)$ and if $F \in CO-CARD$ then $F(X)(Y) = F(E)(X' \cup Y)$.

Sortally reducible quantifiers have a more general property $FIDQ$ or the property of freely increasable domain of quantification. It is defined in D4 and proposition 0 shows that members of INT and of $CO-INT$ have this property:

D4: $F \in FIDQ$ iff there is a binary Boolean function h such that $F(X)(Y) = F(X_1)(h(X_1, Y))$, for any X_1 such that $X \subseteq X_1$.
Proposition 0: $INT \cup CO - INT \subseteq FIDQ$

Since $SOME$ is an intersective quantifier it follows from the above propositons that $SOME$ is a member of $FIDQ$ and thus that *Some wild cats are dangerous* is equivalent to *Some cats are wild and dangerous*.

The algebra INT contains a sub-algebra $CARD$ of cardinal functions: they are denotations of, roughly speaking, various numerals. By definition:

D4: $F \in CARD$ iff for iff for all properties X, Y, W and Z, if $|X \cap Y| = |Z \cap W|$ then $F(X)(Y)$ is true iff $F(W)(Z)$ is true.

Atoms of $CARD$ are functions f_α, such that $f_\alpha(X)(Y) = 1$ iff $|X \cap Y| = \alpha$, for α a cardinal,. Thus the determiner *exactly n* denotes an atomic cardinal function.

As might be expected the algebra $CO - INT$ has an analogous sub-algebra. This is the algebra $CO - CARD$ of co-cardinal functions:

D5: $F \in CO - CARD$ iff for iff for all properties X, Y, W and Z, if $|X - Y| = |W - Z|$ then $F(X)(Y) = F(W)(Z)$

Determiners like *every...except five* denote co-cardinal functions.

All the algebras presented above form the basic stock of algebras for unary NL determiners (cf.[12]). We will study now some other semantic properties of determiners and denotational algebras determined by them.

3 Other Algebras

Let me start the presentation of other algebras by a somewhat less known algebra $GCARD$ of generalised cardinals. It was introduced in [19] in order to account for some properties of the definite article *the*. By definition:

D6: $F \in GCARD$ iff for all properties X, Y, Z if $|X \cap Y| = |X \cap Z|$ then $F(X)(Y) = F(X)(Z)$.

Obviously the algebra $GCARD1$ is a proper sub-algebra of $CONS1$ and contains as proper sub-algebras $CARD$ and $CO - CARD$. Furthermore we have:

Proposition 1: The algebra $GCARD$ is atomic and its atoms are determined as follows: for any $n \leq |E|$ and any $A \subseteq E$, the function $at_{A,n}$ such that $at_{A,n}(X)(Y) = 1$ iff $X = A$ and $|X \cap Y| = n$ is an atom of $GCARD$.

Other properties of $GCARD1$ relate it to other algebras I will introduce now. If $PDET$ is the algebra of possible denotations of determiners, that is all functions in $[P(E) \rightarrow [P(E) \rightarrow \{0,1\}]]$ then conservative functions form a sub-algebra of $PDET$. There are obviously many other sub-algebras of $PDET$. Let us have a look at some of them. First we have a rather natural property of symmetry determining symmetric determiners:

D7: $F \in SYM$ iff for all properties X, Y one has $F(X)(Y) = F(Y)(X)$

We notice that all intersective functions are symmetric though there are symmetric ones which are not intersective. An analogous property which all co-intersective functions have is the property $CONTR$ of being contrapositional:

D8: $F \in CONTR$ iff for all properties X, Y one has $F(X)(Y) = F(Y')(X')$.

The quantifier ALL is obviously contrapositional. Moreover we have:

Proposition 2: $CONS1 \cap SYM = INT$

Proposition 3: $CONS1 \cap CONTR = CO - INT$

We prove proposition 3, more precisely its "from left to right" part. Suppose that for some arbitrary X, Y, W and Z we have (1) $X - Y = W - Z$. This is equivalent to (2) $E \cap (X' \cup Y) = E \cap (W' \cup Z)$. Then:

$F(X)(Y) = F(X)(Y \cup X') =$	- $CONS1$ (fact 2)
$=F(X - Y)(X') =$	- CONTR
$=F(X - Y)(\emptyset) =$	- $CONS1$ (fact 1)
$=F(E)(X' \cup Y) =$	- CONTR
$=F(E)(W' \cup Z) =$	- $CONS1$ (D 1 and (2))
$=F(W - Z)(\emptyset) =$	- CONTR
$=F(W - Z)(W - Z \cap W') =$	- set theoretical equivalence

$=F(W-Z)(W')=$ -$CONS1$ (fact 1)
$=F(W)(W'\cup Z)=$ -CONTR
$=F(W)(Z)$

Thus all co-intersective functions are contrapositional.

There are various properties relating the algebra $GCARD$ to $CARD$ and $CO-CARD$ in a very similar way in which the algebra $CONS1$ is related to INT and $CO-INT$. In particular we have:

Proposition 4: $GCARD1 \cap SYM = CARD$

Proposition 5: $GCARD1 \cap CONTR = CO-CARD$

The proofs of propositions 4 and 5 are similar to the proof of proposition 3.

Propositions 4 and 5 can be used to give a new characterisation of *logical* quantifiers: these are quantifiers which are classically conservative, satisfy the condition of extension and are permutation invariant. Keenan and Westerstahl ([12]) indicate that logical quantifiers are Boolean combinations of cardinal and co-cardinal quantifiers. Given propositions 4 and 5 we have the following characterisation of logical quantifiers (cf. [20]):

Proposition 6: A logical type $\langle 1,1\rangle$ quantifier is a Boolean combination of symmetric generalised cardinals and contrapositional generalised cardinals.

Recent research on natural language type $\langle 1,1\rangle$ quantifiers shows that in fact there are natural classes of such quantifiers which need not to be conservative in the classical sense. In particular such quantifiers can naturally occur in existential contexts (cf. [7]). Similarly Zuber ([17]) strongly suggests that in some languages there are non-conservative determiners which are, however, systematically related to specific conservative ones: they are related by the relation of argument inversion. Let Q be a type $\langle 1,1\rangle$ quantifier. Then:

D 10: Q^i is the inverse of Q iff $Q^i(X)(Y) = Q(Y)(X)$

For instance the determiner *apart from Leo only...* denotes a quantifier which is the inverse of the quantifier denoted by *every...except Leo*. One observes also that symmetric quantifiers are their own inverses. Similarly, inversion preserves contraposition. On the other hand inversion does not preserve (classical) conservativity. It is interesting that inversion of classically conservative quantifiers gives rise to a special class called $CONS2$ ([7]):

D 11: $F \in CONS2$ iff for all properties X, Y, Z if $X \cap Z = Y \cap Z$ then $F(X)(Z) = F(Y)(Z)$

The following fact is obvious now:

Fact 3: $F \in CONS1$ iff $F^i \in CONS2$

Consider the expression *mostly* as a nominal determiner. In this case it denotes the quantifier $MOSTLY$ such that $MOSTLY(X)(Y) = 1$ iff $|Y \cap X| > |Y - X|$.

This means that $MOSTLY$ is the inverse of $MOST$ and thus *Mostly Germans are bear drinkers* is equivalent to *Most bear drinkers are German.*

As with classical conservativity quantifiers satisfying $CONS2$ can be characterised in two other equivalent ways. thus we have:

Fact 4: $F \in CONS2$ iff for any X, Y one has $F(X)(Y) = F(X \cap Y)(Y)$

Fact 5: $F \in CONS2$ iff for any X, Y one has $F(X)(Y) = F(X \cup Y')(Y)$

It is easy to prove proposition 6 from which, in conjunction with proposition 2, follows proposition 7 (stated in Keenan [7]):

Proposition 7: $CONS1 \cap CONS2 \subseteq SYM$

Proposition 8: $CONS1 \cap CONS2 = INT$

In order to obtain a similar result for co-intersective quantifiers we need:

D 12: $F \in CO - CONS2$ iff for all properties X, Y, Z if $X - Z = Y - Z$ then $F(X)(Z) = F(Y)(Z)$

We have the following facts and propositions concerning $CO - CONS2$. We prove Proposition 9, the proof of Proposition 10 being similar:

Fact 6: $F \in CO - CONS2$ iff for any property X, Y one has $F(X)(Y) = F(X - Y)(Y)$

Fact 7: $F \in CO - CONS2$ iff for any property X, Y one has $F(X)(Y) = F(X \cup Y)(Y)$

Proposition 9: $CONS1 \cap CO - CONS2 \subseteq CONTR$

Proposition 10: $CONS1 \cap CO - CONS2 = CO - INT$

Proof of Proposition 9: Suppose $F \in CONS1 \cap CO - CONS2$. Then
$F(X)(Y) = F(X - Y)(Y) =$ - fact 5
$=F(X - Y)(X' \cup Y) =$ - classical conservativity (fact 2)
$=F((X' \cup Y)' \cap Y')(X' \cup Y) =$ - set theoretical equivalence
$=F(Y')(X' \cup Y) =$ - fact 5
$=F(Y')(X')$ - fact 2

Given that co-intersective functions are classically conservative and contrapositive we have to prove the "from left to right" part. So suppose that for some arbitrary sets X, Y, W and Z we have $X - Y = W - Z$ and $F \in CONS1 \cap CO - CONS2$. Then:

$F(X)(Y) = F(X)(X' \cup Y) =$ - fact 2
$=F(X \cap Y')(X') =$ - proposition 9
$=F(X \cap Y')(\emptyset) =$ - $CONS1$ (fact 1)
$=F(W - Z)(W \cap Z' \cap W') =$ - supposition and a set theoretical equivalence

$=F(W-Z)(W') =$ - classical conservativity
$=F(W)(W' \cup Z) =$ - $CONTR$
$=F(W)(Z)$ -classical conservativity (fact 2)

There are two other algebras I would like to discuss briefly. The first is the algebra $PROPORT$ of *proportional* determiners. By definition ([6]):

D13: $F \in PROPORT$ iff for all properties X, Y, W, Z if $|W| \times |X \cap Y| = |X| \times |W \cap Z|$ then $F(X)(Y) = F(W)(Z)$

Proportional quantifiers have the following properties:

Fact 8: Proportional quantifiers form a sub-algebra of $GCARD$.

Fact 9: $F \in PROPORT$ iff $F - not \in PROPORT$

Proposition 11: For $1 \leq m < n$ the functions $F_{m,n}$ such that $F_{m,n}(X)(Y) = 1$ iff $|X \cap Y|/|X| = n/m$ are atoms of $PROPORT$

There are various examples of proportional determiners one of the best known being *most*, in the sense *more than half.* The quantifiers denoted by the determiners like *exactly n percent* are atoms of $PROPORT$. Other examples of proportional quantifiers include *six out of twelve, exactly 20 % , all but a tenth*, etc. According to fact 9, postcomplements of proportional quantifiers are also proportional quantifiers.

It follows from the fact 8 that the class $PROPORT$ includes various *improperly* proportional quantifiers. For instance *less than zero percent, at least zero percent, zero percent, more than zero percent, hundred percent, more than fifty and less than hundred percent*, etc. are (denote), according to D13, proportional quantifiers. These quantifiers correspond to the zero, the unit element of the algebra $PROPORT$, and the quantifiers $SOME$, NO, ALL and $MOST\ BUT\ NOT\ ALL$, respectively. We know that these quantifiers belong also to other algebras. For instance $SOME$ and NO are at the same time cardinal, intersective, symmetric, generalised cardinal and conservative.

In spite of the above examples it is not true in general that cardinal or co-cardinal functions are proportional quantifiers. The exact relationship between $CARD$, $GCARD$ and $PROPORT$ remains to be established. Such a relationship might be helpful to our better understanding of conditions for the first order definissability since proportional quantifiers are precisely known as being in most cases not definable in the first order logic (see [12] for a discussion).

The last algebra I want to mention is the algebra $FPPCPL$ of quantifiers which are fixed points with respect to the operation of postcomplementation:

D 14: $F \in FPPCPL$ iff for all properties X, Y one has $F(X)(Y) = F(X)(Y')$

A natural example of a $FPPCPL$ is given by the denotation of *half of.* The following proposition is a consequence of the fact that postcomplements preserve meets and joins, allows us to obtain other natural examples of such quantifiers:

Proposition 12: For any F type $\langle 1,1\rangle$ we have $F \vee F - not \in FPPCPL$ and $F \wedge F - not \in FPPCPL$

Thus, given that $SOME - not$ equals to $NOT - ALL$ the complex quantifier $SOME\ BUT\ NOT - ALL$ is a fixed point with respect to postcomplement. Similarly a conjunction or a disjunction of an intersective and the corresponding co-intersective forms such a quantifier. This is for instance the case with $FIVE\ AND\ FIVE - not$ or $TEN\ OR\ TEN - not$.

Although the above examples of quantifiers which are fixed points with respect to postcomplements are also examples of (classically) conservative quantifiers, the definition D 14 and Proposition 12 are not restricted to such quantifiers. The following proposition shows that it is possible to define the class $FPPCPL$ in the general unrestricted case using the definitional format adopted here:

Proposition 13: Let $F \in PDET$. Then $F \in FPPCPL$ iff there exists a binary function $\otimes$ on sets for which $X \otimes Y = X \otimes Y'$, for any X, Y, and if $X \otimes Y = X \otimes Z$ then $F(X)(Y) = F(X)(Z)$

Proof $\Rightarrow$. Let $F \in FPPCPL$. Define a binary function "$\otimes$" as follows: $X \otimes Y = F(X)(Y)$. Obviously $X \otimes Y = X \otimes Y'$. It is easy to check that this function is the function we need in order for the conclusion to be satisfied.
$\Leftarrow$ If $\otimes$ is such that $X \otimes Y = X \otimes Y'$ and F satisfies the necessary condition, then $F(X)(Y) = F(X)(Y')$ and thus $F \in FPPCPL$.

Proposition 13 shows how we can define, for most classes of quantifiers we have distinguished, their fixed point sub-classes. What is interesting is the fact that this can be done using the general definitional format adopted here. Thus:

Proposition 14: For any type $\langle 1,1\rangle$ quantifier F, $F \in CONS1 \cap FPPCPL$ iff for any X, Y, Z, $F(X)(Y) = F(X)(Z)$ whenever $X \cap Y = X - Z$.

Similarly we can define, using the following propositions, quantifiers which are proportional, cardinal, co-cardinal, generalised cardinal, etc and at the same time are fixed points with respect to postcomplements. Here are just some such definitional properties illustrating this claim:

Proposition 15: $F \in PROPORT \cap FPPCPL$ iff $F(X)(Y) = F(X)(Z)$ whenever for any X, Y, W, Z, $|W| \times |X \cap Y| = |X| \times |W - Z|$.

Proposition 16: $F \in INT \cap FPPCPL$ iff $F(X)(Y) = F(W)(Z)$ whenever $X \cap Y = W - Z$.

Proposition 17: $F \in GCARD \cap FPPCPL$ iff $F(X)(Y) = F(X)(Z)$ whenever $|X \cap Y| = |X - Z|$.

Proposition 18: $F \in CARD \cap FPPCPL$ iff $F(X)(Y) = F(X)(Z)$ whenever $|X \cap Y| = |W - Z|$.

The following fact limits the usefulness of some of these propositions:

Fact 10: $INT \cap FPPCPL = CO - INT$

This fact indicates that intersective or co-intersective functions which are at the same time fixed points for postcomplement are just the trivial elements of the algebra, the unit and the zero elements. In other words quantifiers which satisfy the condition $F(X)(Y) = F(W)(Z)$ if $X \cap Y = W - Z$ are just the trivial elements of the algebra $PDET$.

Let me give some examples of non-trivial members of $FPPCPL$. They are necessarily neither intersective nor co-intersective. Thus for any k, l, such that $k + l = |E|$ define a type $\langle 1, 1\rangle$ quantifier $F_{k,l}$ as $F_{k,l}(X)(Y) = 1$ iff $|X| = k + l$ and $|X \cap Y| = k$ or $|X| = k+l$ and $|X \cap Y| = l$. This function, which is denoted by the somewhat complex determiner like *Among the (k+l)...exactly k... or exactly l...* is a generalised cardinal function member of $FPPCPL$.

Another example is a determiner, also not very natural, like *either only Leo or else any other...*, (with *either...or* understood as exclusive disjunction) as in the following noun phrase: *either only Leo or else every other student.* This determiner denotes a classically conservative function belonging to $FPPCPL$.

The two preceding examples show the way to obtain proportional quantifiers members of $FPPCPL$: we have just to make disjunctions which in some sense exhaust all possibilities. Thus the determiners like *40 or 60 percent of* or *one third or two thirds of* are determiners denoting such quantifiers.

One of the interests of quantifiers which are fixed points for postcomplements is that they may have linguistic applications showing the usefulness of various generalisations concerning type $\langle 1, 1\rangle$ quantifiers. These quantifiers are considered here as functions taking two sets into truth values. What is essential in most cases is not the fact that such functions have truth-values as their values but the fact that the results of applications of these functions are equal or not to results of application of other functions. So obviously we can generalize most of type $\langle 1, 1\rangle$ quantifiers to functions from pairs of sets onto type of objects other than truth-values. Such a move is useful if we want to analyse "interrogative determiners" in a way similar to the analysis of traditional "declarative" determiners (cf. [4]). Notice now that in questions fixed points may be involved. For instance there is a level of analysis in which the two interrogatives *Which numbers are greater than n ?* and *Which numbers are not greater than n ?* should be considered as being equivalent. Since this equivalence is not the logical equivalence, i.e. not equality of truth-values, a generalisation of involved notions is necessary. This means in particular that the interrogative determiner *which* denotes a generalised function which is fixed with respect to (generalised) postcomplements. The fact that such equivalence need not hold for all interrogatives (cf. [17]) just indicates that there are various types of interrogative determiners.

4 Higher Type Quantifiers

Up to now we were basically interested in the denotations of determiners taking one common noun to form an NP. Binary and, more generally n-ary determiners, have been introduced and extensively studied in [10] and, in somewhat different perspective, in [3]. Their denotations form atomic Boolean algebras. It is interesting that there is a systematic way to form n-ary determiners from Boolean conjunctions and unary determiners ([10]). For instance the complex determiners like *all...and...* or *most ..and...* should be considered as binary determiners because the NP *most students and teachers* does not mean *most individuals who are students and teachers* but rather *most students and most teachers.*

What is important, however, that there exist intrinsically n-ary determiners, that is determiners which are not Boolean compositions of unary ones. In other words we should distinguish between *reducible* and *non-reducible* denotations of n-ary determiners. More precisely, restricting ourselves to binary deterniners and quantifiers of type $\langle\langle 1,1\rangle,1\rangle$ we have (cf. [3]):

D15: F of type $\langle\langle 1,1\rangle,1\rangle$ is $\langle 1,1\rangle$ reducible iff there are two type $\langle 1,1\rangle$ quantifiers F_1 and F_2 and a binary Boolean function h such that for all sets X,Y,Z, $F(X,Y)(Z) = h(F_1(X)(Z), F_2(Y)(Z))$

Clearly the quantifiers $MOST...AND...$ and $ALL...AND...$ are reducible. For reducible quantifiers the following proposition holds:

Proposition 19: Reducible type $\langle\langle 1,1\rangle,1\rangle$ quantifiers form a Boolean algebra.

Beghelli ([3]) shows that comparative determiners like *more...than...* do not denote reducible quantifiers. It follows from proposition 19 that *not more...than...* also denote unreducible quantifiers.

There are thus n-ary determiners denoting higher type quantifiers. We will study only some higher type quantifiers: the ones which correspond to binary relations with relations as arguments. So in general they are of the type $\langle 1^k, 1^l\rangle$ which corresponds to binary relations between k-ary relations and l-ary relations. Even more specifically we will consider quantifiers of this type assuming that either $k = 1$ or $l = 1$. Determiners denoting such quantifiers are sometimes called *structured* (n-ary) determiners.

Most of the properties of type $\langle 1,1\rangle$ quantifiers we studied generalise to higher type quantifiers, if we keep in mind the distinction between "subject" arguments and the "predicate" arguments of a quantifier. The idea, roughly, is that instead of looking at intersections, or cardinalities of the intersection, of the only argument of an unary determiner with the predicative argument we have to look at intersections, or the cardinalities of intersections, of every "subject" argument of the n-ary determiner, with the predicative argument (or predicative arguments). Let us first define the post-complement of a higher type quantifier:

D 16: If F is a type $\langle 1^k, 1^l\rangle$ quantifier then $F - not$ is that type $\langle 1^k, 1^l\rangle$ quantifier for which $F - n(X_1, ..., X_k)(Y_1, ..., Y_l) = F(X_1, ..., X_k)(Y_1', ..., Y_l')$

For other definitions we restrict one of the arguments of the relation to a set. Reducible higher type quantifiers give us a hint as to exactly which intersections should be taken into consideration in such definitions. This is because, as we observe, reducible higher type quantifiers inherit some of their properties from lower order one to which they are reducible. Thus, roughly, if a higher type quantifier is reducible to two conservative quantifiers of lower type then it is reasonable to suppose that the higher type quantifier is conservative. The same with other properties. Having this in mind we get various definitions of higher type quantifiers. In fact some of such definitions have already been applied (see [10],[3],[7]). Thus let D be a type $\langle 1^n, 1\rangle$ quantifier, that is a function from n-tuples of subsets of E to type $\langle 1\rangle$ functions (i.e. denotations of NPs) over E and let $CONS1_{\langle 1^n,1\rangle}$ be the set of (classically) conservative denotations of n-ary structured determiners. Then

D17: $D \in CONS1_{\langle 1^n,1\rangle}$ iff $\forall X_i, Y_1, Y_2$, $D(X_1, ..., X_n)(Y_1) = D(X_1, ..., X_n)(Y_2)$ if $X_i \cap Y_1 = X_i \cap Y_2$, for every $1 \leq i \leq n$.

As in the case of type $\langle 1, 1\rangle$ quantifiers we have equivalent definitions of conservativity for higher type quantifiers. Thus, given that $A_i \cap B = A_i \cap B \cap \bigcup_n A_i$ for all $A_i, B \in E$, we have the following propositions concerning conservativity for higher types:

Proposition 20: $D \in CONS1_{\langle 1^n,1\rangle}$ iff $D(X_1, ..., X_n)(Y) = D(X_1, ..., X_n)(Y \cap \bigcup_n A_i)$

Proposition 21: $D \in CONS1_{\langle 1^n,1\rangle}$ iff $D(X_1, ..., X_n)(Y) = D(X_1, ..., X_n)(Y \cup \bigcap_n A_i')$

In a similar way we define other higher type quantifiers. For instance intersective, co-intersective, cardinal, co-cardinal, and generalised cardinal functions are defined as follows;

D 18: $D \in INT_{\langle 1^n,1\rangle}$ iff $\forall X_i, Y_i, Z_1, Z_2$, $D(X_1, ..., X_n)(Z_1) = D(Y_1, ..., Y_n)(Z_2)$ if $X_i \cap Z_1 = Y_i \cap Z_2$, for every $1 \leq i \leq n$.
D 19: $D \in CO - INT_{\langle 1^n,1\rangle}$ iff whenever $X_i - Z_1 = Y_i - Z_2$, $\forall X_i, Y_i, Z_1, Z_2$, we have $D(X_1, ..., X_n)(Z_1) = D(Y_1, ..., Y_n)(Z_2)$
D20: $D \in CARD_{\langle 1^n,1\rangle}$ iff $\forall X_i, Y_i, Z_1, Z_2$, $D(X_1, ..., X_n)(Z_1) = D(Y_1, ..., Y_n)(Z_2)$ if $|X_i \cap Z_1| = |Y_i \cap Z_2|$, for every $1 \leq i \leq n$.
D21: $D \in CO - CARD_{\langle 1^n,1\rangle}$ iff $\forall X_i, Y_i, Z_1, Z_2$, if $|X_i - Z_1| = |Y_i - Z_2|$, for every $1 \leq i \leq n$, then $D(X_1, ..., X_n)(Z_1) = D(Y_1, ..., Y_n)(Z_2)$
D22: $D \in GCARD_{\langle 1^n,1\rangle}$ iff $\forall X_i, Y_1, Y_2$, $D(X_1, ..., X_n)(Y_1) = D(X_1, ..., X_n)(Y_2)$ if $|X_i \cap Y_1| = |Y_i \cap Y_2|$, for every $1 \leq i \leq n$.

For these quantifiers the following facts hold:

Fact 11: $CONS1_{\langle 1^n,1\rangle}, GCARD_{\langle 1^n,1\rangle}, INT_{\langle 1^n,1\rangle}, CO{-}INT_{\langle 1^n,1\rangle}, CARD_{\langle 1^n,1\rangle}$, $CO - CARD_{\langle 1^n,1\rangle}$ form Boolean algebras.

Fact 12: $CARD_{\langle 1^n,1\rangle} \subseteq INT_{\langle 1^n,1\rangle} \subseteq CONS1_{\langle 1^n,1\rangle}$

Fact 13: $CARD_{\langle 1^n,1\rangle} \cup CO - CARD_{\langle 1^n,1\rangle} \subseteq\ GCARD_{\langle 1^n,1\rangle} \subseteq CONS1_{\langle 1^n,1\rangle}$

In fact more can be shown. Using Proposition 20 and some results from [10] we can indicate atoms of various higher type algebras showing that they are atomic. Thus we have:

Fact 14: Let $1 \leq i \leq n$, $P_i \subseteq E$ and $P \subseteq \bigcup_i P_i$. Then the function $D_{P_1,\ldots,P_n,P}$ such that $D_{P_1,\ldots,P_n,P}(X_1,\ldots,X_n,)(Y) = 1$ iff $X_i = P_i$ and $P = Y \cap \bigcup_i X_i$ is an atom of $CONS1_{\langle 1^n,1\rangle}$. All atoms of $CONS1_{\langle 1^n,1\rangle}$ are of this form.

Fact 15: The function $F_{P_1,\ldots,P_n}$ such that $F_{P_1,\ldots,P_n}(X_1,\ldots,X_n)(Y) = 1$ iff $X_i \cap Y = P_i$ are atoms of $INT_{\langle 1^n,1\rangle}$.

Fact 16: The function $F_{P_1,\ldots,P_n}$ such that $F_{P_1,\ldots,P_n}(X_1,\ldots,X_n)(Y) = 1$ iff $X_i Y' = P_i$ are atoms of $CO - INT_{\langle 1^n,1\rangle}$.

Thus atoms of algebras of higher type intersective and co-intersective functions are determined by sequences of sets. For $n = 1$ we obtain from these facts atoms of the algebras of the corresponding type $\langle 1,1\rangle$ quantifiers. In particular for any $A, B \subseteq E$ such that $B \subseteq A$, the function $F_{A,B}$ such that $F_{A,B}(X)(Y) = 1$ iff $X = A$ and $Y \cap X = B$ is an atom of $CONS1$ (cf. [5]). Furthermore, atoms INT and $CO - INT$ are related to atoms of $CONS1$ in the way indicated in

Proposition 22: Let $F_{P_1,\ldots,P_n,P}$ be an atom of $CONS1_{\langle 1^n,1\rangle}$, $G_{P_1\cap P,\ldots,P_n\cap P}$ be an atom of $INT_{\langle 1^n,1\rangle}$ and $H_{P_1-P,\ldots,P_n-P}$ be an atom of $CO - INT_{\langle 1^n,1\rangle}$. Then $F_{P_1,\ldots,P_n,P} = G_{P_1\cap P,\ldots,P_n\cap P} \wedge H_{P_1-P,\ldots,P_n-P}$.

s

We observe also that the application of the operation of postcomplementation to higher type quantifiers holds results similar to to the ones we get in the case of simple quantifiers. Thus:

Fact 17: $D \in INT_{\langle 1^n,1\rangle}$ iff $D - not \in CO - INT_{\langle 1^n,1\rangle}$.

It is also possible to generalise some other properties studied above to non unary determiners, in particular properties of being symmetric or contrapositional. Recall that quantifiers are relations between relations. Since some of these relations are binary they can be symmetric. Furthermore, since such relations relate Boolean objects, to check whether they relate permuted complements of their arguments is possible as well. This leads to the generalised notion of contrapositionality of a quantifier as well, especially when the objects standing in the binary relation are of the same type. We know that quantifiers corresponding to binary relations may relate objects of different types. For instance, type $\langle 1,1\rangle, 1\rangle$ quantifiers are relations between binary relations and sets (as in *more students than teachers are vegetarians* and type $\langle 1, \langle 1,1\rangle\rangle$ quantifiers are relations between sets and binary relations (as in *more vegetarians are students than teachers*). To extend the notion of symmetry and contrapositionality to such cases we need general definitions to be given in the framework we used in this paper:

D 23: A type $\langle 1^n, 1\rangle$ quantifier D is symmetric iff there exists a binary commutative function $\otimes$ such that $\forall X_i, Y_i, Z_1, Z_2$, $D(X_1, ..., X_n)(Z_1) = D(Y_1, ..., Y_n)(Z_2)$ if $X_i \otimes Z_1 = Y_i \otimes Z_2$, for every $1 \leq i \leq n$.
D 24: A type $\langle 1^n, 1\rangle$ quantifier D is contrapositional iff there exists a binary commutative function $\otimes$ such that if $X_i \otimes Z_1' = Y_i \otimes Z_2'$, $\forall X_i, Y_i, Z_1, Z_2$, all $1 \leq i \leq n$ then $D(X_1, ..., X_n)(Z_1) = D(Y_1, ..., Y_n)(Z_2)$

It is easy to check that if $n = 1$ we get from D 23 the "ordinary" symmetry for simple quantifiers and from D 24 an "ordinary" contraposition for simple quantifiers. For instance one observes that post-complements of symmetric quantifiers are contrapositional and *vice versa*. Furthermore the following propositions are true:

Proposition 23: Let $F \in PDET_{\langle 1^n,1\rangle}$,$G \in PDET_{\langle 1,1^n\rangle}$ and $F(X_1, ..., X_n)(Y) = G(Y)(X_1, ..., X_n)$. Then F is symmetric iff G is symmetric.

Proposition 24: Let $F \in PDET_{\langle 1^n,1\rangle}$, $G \in PDET_{\langle 1,1^n\rangle}$ and $F(X_1, ..., X_n)(Y) = G(Y')(X_1', ..., X_n')$. Then F is contrapositional iff G is contrapositional.

Proportional n-ary determiners represent a more difficult case since proportionality in this case may depend on whether we consider reducible or unreducible quantifiers. We have seen that determiners like *most* or *20 percent of* denote proportional type $\langle 1, 1\rangle$ quantifiers. The question we can ask now is the following: are the reducible type $\langle\langle 1, 1\rangle, 1\rangle$ quantifiers based on them and the corresponding type $\langle 1\rangle$ quantifiers also proportional? Should we consider for instance that the NP *most students and teachers* (considered as containing a binary determiner) involves proportional quantification. I think the answer should be negative. The reason is that in this case there is no proportional relation between the two arguments: *students* and *teachers*. The truth value of the sentence *Most students and teachers danced* does not depend on any proportional relation between the number of students and the number of teachers. This is different from the case of unreducible determiners like *twice as many students as teachers* since in the latter example there is a (non trivial) proportional relation between the number of students and the number of teachers. Any general definition of proportional quantifiers should account for this difference. This leads to the following definition of type $\langle\langle 1, 1\rangle, 1\rangle$ proportional quantifiers:

D 25: $D \in PROPORT_{\langle 1^2,1\rangle}$ iff for all $X_1, X_2, Y_1, Y_2, Z_1, Z_2$, $D(X_1, X_2)(Z_1) = D(Y_1, Y_2)(Z_2)$ whenever $|Y_1| \times |Y_2| \times |X_1 \cap Z_1| = |X_1| \times |X_2| \times |Y_1 \cap Z_2|$ and $|Y_1| \times |Y_2| \times |X_2 \cap Z_1| = |X_1| \times |X_2| \times |Y_2 \cap Z_2|$

One checks by calculation that according to D 22 determiners like *twice as many... as...* or *20 percent more...than...* denote proportional quantifiers whereas the reducible quantifier $MOST...AND...$ is not proportional. Furthermore the following proposition is true:

Proposition 25: $PROPORT_{\langle 1^2,1\rangle}$ is a sub-algebra of $GCARD_{\langle 1^2,1\rangle}$

Incidentally the analogue of fact 9 (that proportionality is closed with respect to postcomplements) does not hold for the higher type quantifiers. We observe also that even if definition D 25 can be formally extended to n-ary cases it is not clear what would be empirical content of such extensions. In fact it seems that proportional quantifiers still need to be studied more deeply.

5 Conclusive Remarks

Most of this paper is occupied by the presentation of some new or more general algebras representing possible denotations for nominal determiners, not only unary ones. I left aside the problem of how many quantifiers there are in various algebras when different constraints defining them are taken into account. The exact number of quantifiers, members of a given fixed algebra, depends of course on the number of individuals in the model and a specific definitional restriction. Some of these numbers have been established for various specific cases ([15], [10], [2], [6], [12], [8]).

The algebras which were studied here are basically $GCARD$, SYM, $CONTR$, $FPPCPL$ and, to some extend, $PROPORT$. One of the reasons for introducing them is that they give rise to natural inferential patterns. Another reason is that all of them can be considered, it seems to me, as representing "borderline cases" between "linguistically useful" and "logically natural" quantifiers. For instance, concerning SYM, $CONTR$ and $FPPCPL$ we observe that they are defined by very natural "logical" properties. The naturalness of these properties and relations between SYM and INT on the one hand and between $CONTR$ and $CO-INT$ on the other hand established above are sufficient reasons, I believe, to study them. Proportional quantifiers are on the border-line because they involve two different cognitive competences: surely a linguistic competence and, in addition, an elementarily arithmetic competence which is probably not a linguistic one.

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