

Symmetric and contrapositional quantifiers *

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1. Introduction

In his pioneering work Mostowski (1957) made the very insightful suggestion that quantifiers are special higher order relations, relations between relations (where sets are unary relations). This move led eventually to the generalised quantifier theory (GQT). In particular it made it possible to study "classical" quantifiers using familiar tools of the elementary theory of relations. This simple observation gave rise to various new results basically, but not exclusively, in the domain of natural language semantics (Barwise and Cooper 1981, van Benthem 1986, Keenan and Stavi 1986, Keenan 1993, Keenan and Westerstahl 1997, Zuber 2005). Furthermore, since arguments of relations are in this case Boolean objects, the combinatorics of possibly related arguments is much richer. Thus we can see what happens not only when we commute arguments of a relation but also when in addition we replace these arguments by their various Boolean compounds. The purpose of this paper is to explore one such possibility and study in particular a special class of quantifiers, called *contrapositional* that is quantifiers Q corresponding to a binary relation between sets such that $Q(X)(Y) = Q(Y')(X')$, where X' is the Boolean complement of X . We will see that contrapositional quantifiers are closely related to symmetric quantifiers, that is quantifiers corresponding to symmetric relations.

The framework which will be used to study symmetric and contrapositional quantifiers is that of Boolean semantics (Keenan and Faltz 1986). One of the advantages of this approach is that it allows us to use a general definitional format which makes it easier to see Boolean structure of analysed objects and permits simple comparisons and generalisations in many cases. I will study symmetry and contraposition of quantifiers corresponding to binary relations between sets, formally type $\langle 1, 1 \rangle$ quantifiers, generically called here *simple quantifiers*. Other quantifiers, called here *higher type quantifiers*, which are quantifiers corresponding to binary relations over n -ary relations, $n > 1$, will also

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be considered. The basic result in this case concerns the generalisation of the notion of symmetry and contraposition to binary relations whose two arguments have different types. In many cases I will illustrate various definitions and results by English examples but it will very quickly become obvious that such an illustration must be very partial.

In the next section I recall various definitions and properties of simple quantifiers. These properties are useful for the analysis of symmetric and contrapositional quantifiers that is given in section 3. Then, in section 4, some generalisations and extensions to the case of higher type quantifiers are provided.

Most of the results presented in this paper are not technically complex and will be given without proofs as obvious facts. Although many results obtained are also valid for infinite universes I will implicitly assume their finiteness.

2. Simple quantifiers

In GQT quantifiers are, roughly speaking, denotations of NPs or of their syntactic parts. Formally they are higher order relations. They can be expressed as functions onto truth-values. For instance quantifiers of type $\langle 1 \rangle$ are functions from $P(E)$, the power set of E , the universe of objects, onto $\{0, 1\}$.

In this section we will be basically interested in the logic of denotations of (unary) nominal determiners. These are expressions (like *every*, *no*, *some...including Lea*, *most*) which combine with common nouns to form noun phrases. Thus, semantically, they are functions from $P(E)$ onto type $\langle 1 \rangle$ quantifiers. They are type $\langle 1, 1 \rangle$ quantifiers and will be called here *simple quantifiers*. These quantifiers can be viewed as binary relations on sets. Indeed a type $\langle 1, 1 \rangle$ quantifier F , which is a function in $[P(E) \rightarrow [P(E) \rightarrow \{0, 1\}]]$ corresponds to the binary relation Q between sets defined by $QXY \Leftrightarrow F(X)(Y) = 1$.

Let us denote the set of all type $\langle 1, 1 \rangle$ quantifiers, or the set of unrestricted functions belonging to $[P(E) \rightarrow [P(E) \rightarrow \{0, 1\}]]$ by $PDET$. Obviously we have:

Fact 0: The set $PDET$ forms a complete atomic Boolean algebra with Boolean operations defined pointwise. Atoms of $PDET$ are the functions determined by two sets defined as follows: for any set A and B the function $F_{A,B}$ such that $F_{A,B}(X)(Y) = 1$ iff $X = A$ and $Y = B$ are atoms of $PDET$. Furthermore, any atom of $PDET$ has this form.

Thus atoms of *PDET* are determined by ordered pairs of sets and any atom of *PDET* determines an ordered pair of sets.

The Boolean character of quantifiers and their arguments allows us to distinguish two types of negations or complements. The first is the usual Boolean complement. The second, called *postcomplement* is defined by the complement of one of its arguments. More precisely, let F be a type $\langle 1, 1 \rangle$ quantifier. Then its postcomplement, noted $F\neg$, is the type $\langle 1, 1 \rangle$ quantifier defined as $F\neg(X)(Y) = F(X)(Y')$. Similarly, if Q is a type $\langle 1 \rangle$ quantifier, then its postcomplement $Q\neg$ is defined as $Q\neg(Y) = Q(Y')$, or, equivalently: $Y \in Q\neg$ iff $Y' \in Q$. Syntactically, postcomplement corresponds to the negation of verb phrases. In what follows we will use the fact that postcomplements of meets or joins of two functions are respectively meets and joins of postcomplements of these functions.

Research on determiners in natural languages shows that their denotations satisfy some specific conditions and constraints. One of the best known such constraints on possible denotations of determiners is conservativity. Since sometimes two different notions of conservativity are used (Keenan 2003), which depend on which specific arguments intersection is taken into consideration, I will refer to this "classical" notion of conservativity as *CONS1*. By definition:

D1: $F \in CONS1$ iff for any property X, Y and Z if $X \cap Y = X \cap Z$ then $F(X)(Y) = F(X)(Z)$

Definition D1 is given in the by now standard format in which most of the following definitions will be given. Classical conservativity can also be formulated in two different ways:

Fact 1 (cf. Keenan and Faltz 1986) $F \in CONS1$ iff for any property X, Y one has $F(X)(Y) = F(X)(X \cap Y)$

Fact 2: $F \in CONS1$ iff for any property X, Y one has $F(X)(Y) = F(X)(X' \cup Y)$

The algebra *CONS1* has two important sub-algebras, the algebra *INT* of intersective functions, and the algebra *CO-INT1* of (classically) co-intersective functions (Keenan 1993). By definition:

D2: $F \in INT$, iff for all properties X, Y, Z and W , if $X \cap Y = Z \cap W$ then $F(X)(Y) = F(Z)(W)$.

D3: $F \in CO-INT1$ iff for all properties X, Y, Z and W , if $X - Y = Z - W$ then $F(X)(Y) = F(Z)(W)$.

Elements of *INT* and of *CO-INT1* have in particular the following properties:

Fact 3: The following three conditions are equivalent: (i) $F \in INT$, (ii) $F(X)(Y) = F(X \cap Y)(X \cap Y)$, (iii) $F(X)(Y) = F(E)(X \cap Y)$

Fact 4: The following four conditions are equivalent: $F \in CO-INT1$, (ii) $F(X)(Y) = F(X - Y)(X' \cup Y)$, (iii) $F(X)(Y) = F(X - Y)(\emptyset)$, (iv) $F(X)(Y) = F(E)(X' \cup Y)$

Both sets, *INT* and *CO-INT1*, form atomic (and complete) Boolean algebras. Their atoms are determined by a property (Keenan 1993): for any property P the function F_P such that $F_P(X)(Y) = 1$ iff $X \cap Y = P$ is an atom of *INT* and the function F_P such that $F_P(X)(Y) = 1$ iff $X - Y = P$ is an atom of *CO-INT*. Exclusion determiners (Zuber 1998) denote such atomic functions: *no...except Leo and Lea* denotes an atom of the algebra of intersective function determined by the set composed of two elements, Leo and Lea.

Algebras *INT* and *CO-INT1* are important for two reasons. First, as shown in Keenan (1993) the algebra *CONS1* is a Boolean closure of *INT* and *CO-INT1*. Second, Keenan (1993) shows also that quantifiers belonging to *INT* or to *CO-INT* are exactly those which are sortally reducible. This means, roughly speaking, that it is possible to replace *salva veritate* the first argument of these quantifiers by the universal property E and replace the second argument by a Boolean combination of the first argument with the second (cf. fact 3 iii and fact 4 iv).

The algebra *INT* contains a sub-algebra *CARD* of (intersective) cardinal determiners: they are denotations of, roughly speaking, various numerals. By definition:

D4: $F \in CARD$ iff for all properties X, Y, W and Z , if $|X \cap Y| = |W \cap Z|$ then $F(X)(Y) = F(W)(Z)$.

Atoms of *CARD* are the functions f_α , where α is a cardinal, such that $f_\alpha(X)(Y)$ is true iff $|X \cap Y| = \alpha$. From this definition it follows that any cardinal determines an atom of *CARD*. Thus the determiner *exactly n* denotes an atomic cardinal function.

As might be expected the algebra *CO-INT1* has an analogous sub-algebra. This is the algebra *CO-CARD* of co-cardinal functions:

D5: $F \in CO-CARD$ iff for all properties X, Y, W and Z , if $|X - Y| = |W - Z|$ then $F(X)(Y) = F(W)(Z)$

Determiners like *every...except five* denote co-cardinal functions.

Of linguistic relevance (cf. Keenan 2003) are functions $CONS2$, which are conservative on the second argument:

D 6: $F \in CONS2$ iff $F(X)(Z) = F(Y)(Z)$ whenever $X \cap Z = Y \cap Z$.

The set $CONS2$ forms an atomic Boolean algebra. It has also many other properties which often mirror the properties of $CONS1$. In particular, as the following fact shows, $CONS2$ can be defined in three different ways:

Fact 5: $F \in CONS2$ iff $F(X)(Y) = F(X \cap Y)(Y)$ iff $F(X \cup Y')(Y)$

Clearly the algebra INT is one of sub-algebras of $CONS2$. The following proposition shows how $CONS2$ is related to $CONS1$ via INT (Keenan 2003):

Proposition 0: $CONS1 \cap CONS2 = INT$.

The algebra $CONS2$ has another important sub-algebra, the algebra $CO-INT2$. It is defined as follows:

D 7: $F \in CO-INT2$ iff $F(X)(Y) = F(W)(Z)$ whenever $Y - X = Z - W$

The algebra $CO-INT2$ is an analogue of $CO-INT1$ for $CONS1$. Obviously $CO-INT1$ and $CO-INT2$ are isomorphic.

The last algebra for determiners I need to mention is a less known algebra of generalised cardinals or $GCARD$. It was introduced in Zuber 2004 in order to account for some properties of the definite article *the*. By definition:

D8 $F \in GCARD$ iff for all properties X, Y, Z if $|X \cap Y| = |X \cap Z|$ then $F(X)(Y) = F(X)(Z)$.

Obviously the algebra $GCARD$ is a proper sub-algebra of $CONS1$ and contains as proper sub-algebras $CARD$ and $CO-CARD$. Furthermore we have:

Proposition 1: The algebra $GCARD$ is atomic and its atoms are determined as follows: for any $n \leq |E|$ and any $A \subseteq E$ the function $at_{A,n}$ such that $at_{A,n}(X)(Y) = 1$ iff $X = A$ and $|X \cap Y| = n$ is an atom of $GCARD$.

In next sections many applications of $GCARD$ will be shown.

3. Simple symmetric and contrapositional quantifiers

We are already well acquainted with the restriction of symmetry as applied to type $\langle 1, 1 \rangle$ quantifiers:

D9: $F \in SYM$ iff for all properties X, Y one has $F(X)(Y) = F(Y)(X)$

Strictly speaking the equality (equivalence) in D 9 can be replaced by an entailment.

The quantifier F defined as $F(X)(Y) = 1$ iff $X \subseteq Y$ or $Y \subseteq X$ is symmetric. We obtain more interesting examples of symmetric quantifiers due to the following observation:

Fact 6: INT is a sub-algebra of SYM

For example *SOME...*, *INCLUDING LEA* and *NO, ..., EXCEPT LEO* are symmetric quantifiers.

Given that "ordinary" symmetric relations form an atomic Boolean algebra we have the following fact concerning symmetric quantifiers:

Fact 7: SYM forms an atomic Boolean algebra, a sub-algebra of $PDET$, with Boolean operations defined pointwise. For any set A and B the function $F_{A,B}$ defined as $F_{A,B}(X)(Y) = 1$ if $X = A$ and $Y = B$ or if $X = B$ and $Y = A$ and $F_{A,B}(X)(Y) = 0$ otherwise, is an atom of SYM .

Thus any atom $F_{A,B}$ of SYM is determined by a pair $\langle A, B \rangle$ of sets and any atom of SYM determines a pair of sets.

It is interesting that one can define symmetric quantifiers in the format we use in other definitions and which can be easily generalised to quantifiers of higher types. The following proposition which indicates such an equivalent definition, although trivial, is proved since it will be used as a convenient handle for generalising symmetry to higher types :

Proposition 2: $F \in SYM$ iff there exists a commutative binary function " \otimes " taking sets as arguments such that for all X, Y, W, Z if $X \otimes Y = W \otimes Z$ then $F(X)(Y) = F(W)(Z)$.

Proof \Rightarrow . Let $F \in SYM$. Define a binary function " \otimes " from pairs of sets to truth-values as follows: $X \otimes Y = F(X)(Y)$. Obviously \otimes is commutative. It is easy to check that this function is the function we need in order for the conclusion to be satisfied.

\Leftarrow If \otimes is commutative then $X \otimes Y = Y \otimes X$. This entails, given the necessary condition, that $F(X)(Y) = F(Y)(X)$ and thus F is

symmetric.

We can thus give various sufficient conditions for symmetry without explicitly indicating the permutation of arguments in the quantifier. We will use this fact to define symmetry of quantifiers of higher types. Of course symmetric quantifiers which can be obtained in the way indicated by proposition 2 need not to be conservative. For instance the quantifier F such that $F(X)(Y) = 1$ iff $|X| = |Y|$, although symmetric, is not conservative.

The next proposition indicates the precise relationship between symmetric quantifiers in general and quantifiers which are symmetric and classically conservative:

Proposition 3: $CONS1 \cap SYM = INT$

Thus intersective quantifiers are exactly those which are conservative and symmetric.

We have already seen in the introduction how to define contrapositional quantifiers. Thus, to repeat:

D10: $F \in CONTR$ iff for all sets X, Y one has $F(X)(Y) = F(Y')(X')$.

We can replace the equivalence in D 10 by an entailment without changing the content of the definition.

The first general property of contrapositional quantifiers is indicated in:

Fact 8: $CONTR$ forms an atomic Boolean algebra, a sub-algebra of $PDET$, with Boolean operations defined pointwise. For any set A and B the function $F_{A,B}$ defined as $F_{A,B}(X)(Y) = 1$ if $X = A$ and $Y = B$ or if $X = B'$ and $Y = A'$ and $F_{A,B}(X)(Y) = 0$ otherwise, is an atom of $CONTR$

As in the case of symmetry there is an equivalent way of defining contrapositional quantifiers. Thus

Proposition 4: $F \in CONTR$ iff there exists a commutative binary function " \otimes " taking sets as arguments such that for all X, Y, W, Z if $X \otimes Y' = W \otimes Z'$ then $F(X)(Y) = F(W)(Z)$.

It is possible to give various examples of non-trivial members of $PDET$ which are contrapositional and symmetric at the same time. As we will see they cannot be classically conservative, however. Further-

more, one observes that contrapositional quantifiers are systematically related to symmetric ones. In particular we obtain examples of contrapositional quantifiers as a consequence of:

Fact 9: $F \in SYM$ iff $F-n \in CONTR$

We know that the quantifier F defined as $F(X)(Y) = 1$ iff $|X| = |Y|$ is symmetric. It follows from fact 9 that the quantifier G defined as $G(X)(Y) = 1$ iff $|X| = |Y'|$ is contrapositional.

As indicated above not all contrapositional quantifiers are classically conservative. For instance the quantifier F such that $F(X)(Y) = F(X \cup Y')(X' \cap Y)$ although contrapositional is not classically conservative. It belongs, however, to the class $CONS2$. Thus we have:

Fact 10: $CO-INT1$ and $CO-INT2$ are sub-algebras of $CONTR$.

Fact 10 indicates that all co-intersective functions, those from $CO-INT1$ and those from $CO-INT2$ are contrapositional. For instance *Every student except Leo is vegetarian* is equivalent to *Every non vegetarian except Leo is not a student*.

The algebras SYM and $CONTR$ are important sub-algebras of $PDET$ in a similar way in which the algebras INT and $CO-INT$ are important sub-algebras of $CONS1$. More precisely any element of $PDET$ is a Boolean combination of elements of SYM and of $CONTR$. To see this notice first:

Fact 11: If $F_{A,B}$ is an atom of $PDET$, determined by the pair A, B of sets, then $F_{A,B} = G_{A,B} \wedge H_{B',A'}$, where $G_{A,B}$ is the atom of SYM which is determined by $\langle A, B \rangle$ and $H_{B',A'}$ is the atom of $CONTR$ determined by the pair $\langle B', A' \rangle$

Contrapositional quantifiers are related to conservative and co-intersective quantifiers in the same way as symmetric ones are related to conservative and intersective quantifiers. This is shown in:

Proposition 5: $CONS1 \cap CONTR = CO-INT1$

We prove "from left to right" part. Suppose that for some arbitrary X, Y, W and Z we have (1) $X - Y = W - Z$. This is equivalent to (2) $E \cap (X' \cup Y) = E \cap (W' \cup Z)$. Then:

$$\begin{aligned} F(X)(Y) &= F(X)(Y \cup X') = && - CONS1 \text{ (fact 2)} \\ &= F(X - Y)(X') = && - CONTR \end{aligned}$$

$$\begin{aligned}
&=F(X - Y)(\emptyset) = && - CONS1 (fact 1) \\
&=F(E)(X' \cup Y) = && - CONTR \\
&=F(E)(W' \cup Z) = && - CONS1 (D 1 and (2)) \\
&=F(W - Z)(\emptyset) = && - CONTR \\
&=F(W - Z)(W - Z \cap W') = && - set theoretical equivalence \\
&=F(W - Z)(W') = && -CONS1 (fact 1) \\
&=F(W)(W' \cup Z) = && - CONTR \\
&=F(W)(Z)
\end{aligned}$$

Quite naturally we have similar relations between algebras *GCARD*, *CARD*, *CO-CARD* and *SYM* and *CONTR*. More precisely, the following propositions hold:

Proposition 6: $GCARD \cap SYM = CARD$

Proposition 7: $GCARD \cap CONTR = CO-CARD$

Proof of Proposition 6: We have to prove only the "from left to right" part. Suppose that $F \in GCARD \cap SYM$. This means, according to propositions 1 and 3 that F is intersective. Suppose now that for some arbitrary X, Y, W, Z we have (i) $|X \cap Y| = |W \cap Z|$. This equality is equivalent to (ii) $|E \cap (X \cap Y)| = |E \cap (W \cap Z)|$. Then:

$$\begin{aligned}
F(X)(Y) &= F(E)(X \cap Y) && - F \text{ is intersective (and Fact 3)} \\
&= F(E)(W \cap Z) && - F \in GCARD \text{ and equality (ii)} \\
&= F(W)(Z) && - F \text{ is intersective (and Fact 3)}
\end{aligned}$$

Using fact 4 we prove in the similar way proposition 7. Furthermore using fact 5 we can prove the following proposition:

Proposition 8: $CONS2 \cap CONTR = CO-INT2$

The last series of properties of *SYM* and *CONTR* that I want to indicate can be expressed by the notion of the *inverse* of a given quantifier. Thus:

D 11: Let Q be a type $\langle 1, 1 \rangle$ quantifier. Then Q^i is the inverse of Q iff $Q^i(X)(Y) = Q(Y)(X)$

For instance the determiner *apart from Leo only...* denotes a quantifier which is the inverse of the quantifier denoted by *every...except Leo*. More generally one observes for instance that $F \in CONS1$ iff $F^i \in CONS2$ and $F \in CO-INT1$ iff $F^i \in CO-INT2$. Now concerning the symmetric and contrapositional quantifiers the following is true:

Fact 12: $F \in SYM$ iff $F^i \in SYM$ and $F \in CONTR$ iff $F^i \in CONTR$

We know that members of *CO-INT1* are contrapositional. Their inverses are members of *CO-INT2*. The inverse of *EVERY...EXCEPT LEO* (which is a member of *CO-INT1*) is *APART FROM LEO ONLY...* (a member of *CO-INT2*). Since the former is contrapositional, according to Fact 12 the latter is also contrapositional. Thus *Apart from Leo only students are vegetarians* is equivalent to *Apart from Leo only non-vegetarians are non-students*.

4. Quantifiers of higher types

In the previous section we discussed quantifiers of type $\langle 1,1 \rangle$ which correspond to binary relations between sets. We want now to consider a more general case of quantifiers which are still binary relations but arguments of these relations are other relations, usually binary, or sets and binary relations. We will see that even if from a formal point of view we need not impose any limitation on the arity of arguments of generalised quantifiers, for empirical reasons we will often limit ourselves to quantifiers having at most binary relations as arguments.

Let us first see an example. Often discussed in the literature on "natural" higher order quantifiers are so-called *comparative binary* quantifiers (Keenan and Moss 1984, Beghelli 1992). These are quantifiers denoted by discontinuous (binary) determiners like *More... than* or *As many... as*. When these determiners form subject NPs (in sentences with "simple" VPs) then they denote quantifiers of type $\langle \langle 1,1 \rangle 1 \rangle$. These quantifiers can be said to be genuine higher order since they cannot be reduced to a Boolean combination of simple quantifiers (Beghelli 1994). Furthermore, they are "natural" in the sense that the determiners by which they are denoted have a categorial status of syntactically justified binary determiners (Keenan 1989).

Obviously some higher type quantifiers pose from the beginning a special problem for two basic properties we study here: how should we define symmetry and contraposition in the general case. For indeed, if the type of a function or a relation depends on the type of its different arguments then a binary relation which has as first argument an n-ary relation and as the second argument an m-ary relation is not the same as a binary relation which has as its first argument an m-ary relation and as the second argument an n-ary relation. But this just means that we cannot define the symmetry or contraposition just by comparing relations with inverted or permuted arguments because the inversion of arguments changes the type of relation. So we have two

introductory tasks: to extend definitions of properties like conservativity, intersectivity, etc. to quantifiers of higher types and define for such quantifiers the notions of symmetry and contraposition.

Some of the definitions we are looking for have already been suggested (cf. Keenan and Moss 1984, Beghelli 1994, Keenan 2003, Zuber 2005). After the discussion from the previous section we have a relatively clear intuition of how to define conservative quantifiers of higher types. Here are two general definitions: the definition of conservative quantifiers and the definition of straightforwardly related generalised cardinals (Zuber 2004, 2005):

D12: $D \in CONS1_{\langle 1^n, 1 \rangle}$ iff $\forall X_i, Y_1, Y_2$, if $X_i \cap Y_1 = X_i \cap Y_2$, for every $1 \leq i \leq n$ then $D(X_1, \dots, X_n)(Y_1) = D(X_1, \dots, X_n)(Y_2)$.

D13: $D \in GCARD_{\langle 1^n, 1 \rangle}$ iff $\forall X_i, Y_1, Y_2$, if $|X_i \cap Y_1| = |X_i \cap Y_2|$, for every $1 \leq i \leq n$ then $D(X_1, \dots, X_n)(Y_1) = D(X_1, \dots, X_n)(Y_2)$.

Obviously we have:

Fact 13: $GCARD_{\langle 1^n, 1 \rangle} \subset CONS1_{\langle 1^n, 1 \rangle}$

Comparing the above definitions with the definitions of intersective, co-intersective, cardinal and co-cardinal simple quantifiers we see how to define higher order intersective, co-intersective, cardinal and co-cardinal quantifiers of type $\langle 1^n, 1 \rangle$. Thus we have the following definitions:

D 14: $D \in INT_{\langle 1^n, 1 \rangle}$ iff $\forall X_i, Y_i, Z_1, Z_2$, if $X_i \cap Z_1 = Y_i \cap Z_2$ then $D(X_1, \dots, X_n)(Z_1) = D(Y_1, \dots, Y_n)(Z_2)$, for every $1 \leq i \leq n$.

D 15: $D \in CO-INT_{\langle 1^n, 1 \rangle}$ iff $\forall X_i, Y_i, Z_1, Z_2$, if $X_i - Z_1 = Y_i - Z_2$, for every $1 \leq i \leq n$ then $D(X_1, \dots, X_n)(Z_1) = D(Y_1, \dots, Y_n)(Z_2)$

D16: $D \in CARD_{\langle 1^n, 1 \rangle}$ iff $\forall X_i, Y_i, Z_1, Z_2$, if $|X_i \cap Z_1| = |Y_i \cap Z_2|$, for every $1 \leq i \leq n$ then $D(X_1, \dots, X_n)(Z_1) = D(Y_1, \dots, Y_n)(Z_2)$

D17: $D \in CO-CARD_{\langle 1^n, 1 \rangle}$ iff $\forall X_i, Y_i, Z_1, Z_2$, $D(X_1, \dots, X_n)(Z_1) = D(Y_1, \dots, Y_n)(Z_2)$ if $|X_i - Z_1| = |Y_i - Z_2|$, for every $1 \leq i \leq n$.

Various classes of quantifiers specified in these definitions are related between themselves. We have in particular (cf. Keenan and Moss 1984):

Fact 14: $CONS1_{\langle 1^n, 1 \rangle}, GCARD_{\langle 1^n, 1 \rangle}, INT_{\langle 1^n, 1 \rangle}, CO-INT_{\langle 1^n, 1 \rangle}, CARD_{\langle 1^n, 1 \rangle}, CO-CARD_{\langle 1^n, 1 \rangle}$ form Boolean algebras.

Fact 15: $CARD_{\langle 1^n, 1 \rangle} \subseteq INT_{\langle 1^n, 1 \rangle} \subseteq CONS1_{\langle 1^n, 1 \rangle}$

Another easy to establish relation (holding in finite models), and analogous to that established in the previous section of quantifiers of type $\langle 1, 1 \rangle$ is indicated in:

Fact 16: $CARD_{\langle 1^n, 1 \rangle} \cup CO-CARD_{\langle 1^n, 1 \rangle} \subseteq GCARD_{\langle 1^n, 1 \rangle} \subseteq CONS1_{\langle 1^n, 1 \rangle}$

To be more precise the above set inclusions can in fact be replaced by statements indicating that included sets are sub-algebras of sets in which they are included.

It is easy to check that the quantifier *MORE...THAN...* denoted by the binary determiner *More ...than...* (occurring in NPs on the subject position) is a type $\langle 1^2, 1 \rangle$ cardinal quantifier. Other definitions given above can be illustrated by linguistically less natural examples of complex quantifiers obtained by the interaction of simple quantifiers with various conjunctions. Recall (Keenan and Moss 1984) that natural languages have a very productive way of forming higher type quantifiers using specific quantifiers with n-ary conjunctions. This possibility is based on the observation that very often a conjunction in the scope of a type $\langle 1, 1 \rangle$ quantifier gives rise to "distributive readings". Thus the NP *most students and teachers* does not mean *most individuals who are students and teachers* but rather *most students and most teachers*. This is tantamount to saying that the determiner *most ...and...* denotes a type $\langle \langle 1, 1 \rangle, 1 \rangle$ quantifier. This fact is sufficient to provide various examples of higher type quantifiers. Notice in addition that this way of augmenting arities by augmenting the number of conjuncts works for arities higher than 2: if the conjunction has n-conjuncts then the quantifier obtained in this way is a type $\langle 1^n, 1 \rangle$ quantifier. Given these observations it is easy to check that *SOME...AND...* is intersective, *FIVE...AND...* is cardinal, *EVERY...AND...* is co-cardinal (but also co-intersective). All of them are type $\langle \langle 1, 1 \rangle, 1 \rangle$ quantifiers (if they take as the third (predicative) argument a simple (non conjoined) set). For similar reasons we note that the quantifier *MOST...AND...AND...* is generalised cardinal of type $\langle \langle 1, 1, 1 \rangle, 1 \rangle$.

All the above definitions specify various classes of type $\langle 1^n, 1 \rangle$ quantifiers. Natural languages have also clear cases of type $\langle 1, 1^n \rangle$ quantifiers. An example of such a quantifier is the comparative quantifier *MORE...ARE...THAN...* (as the denotation of the determiner occurring in *More students are Buddhists than shogi players*. This means that we have to define various classes of type $\langle 1, 1^n \rangle$ quantifiers as well. Here are some relevant definitions:

D 18: $D \in CONS1_{\langle 1, 1^n \rangle}$ iff for all X, Y_i, Z_i , if $X \cap Y_i = X \cap Z_i$, for every $1 \leq i \leq n$ then $D(X)(Y_1, \dots, Y_n) = D(X)(Z_1, \dots, Z_n)$.

- D 19: $D \in GCARD_{\langle 1,1^n \rangle}$ iff for all X, Y_i, Z_i , if $|X \cap Y_i| = |X \cap Z_i|$, for every $1 \leq i \leq n$ then $D(X)(Y_1, \dots, Y_n) = D(X)(Z_1, \dots, Z_n)$.
- D 20: $D \in INT_{\langle 1,1^n \rangle}$ iff for all X_1, X_2, Y_i, Z_i , $D(X_1)(Y_1, \dots, Y_n) = D(X_2)(Z_1, \dots, Z_n)$, whenever $X_1 \cap Y_i = X_2 \cap Z_i$, for every $1 \leq i \leq n$.
- D 21: $D \in CO-INT_{\langle 1,1^n \rangle}$ iff for all X_1, X_2, Y_i, Z_i , $D(X_1)(Y_1, \dots, Y_n) = D(X_2)(Z_1, \dots, Z_n)$, whenever $X_1 - Y_i = X_2 - Z_i$, for every $1 \leq i \leq n$.
- D 22: $D \in CARD_{\langle 1,1^n \rangle}$ iff for all X_1, X_2, Y_i, Z_i , $D(X_1)(Y_1, \dots, Y_n) = D(X_2)(Z_1, \dots, Z_n)$, whenever $|X_1 \cap Y_i| = |X_2 \cap Z_i|$, for every $1 \leq i \leq n$.
- D 23: $D \in CO-CARD_{\langle 1,1^n \rangle}$ iff for all X_1, X_2, Y_i, Z_i , if $|X_1 - Y_i| = |X_2 - Z_i|$ then $D(X_1)(Y_1, \dots, Y_n) = D(X_2)(Z_1, \dots, Z_n)$, for every $1 \leq i \leq n$.

It is easy to see that the quantifier *MORE...ARE...THAN...* (as the denotation of the determiner occurring in *More students are Buddhists than shogi players*) is cardinal, in the sense of the definition D22. Other definitions above can be illustrated by quantifiers which are denotations of "complex determiners" composed of unary determiners with conjunctions in the predicate. Consider the sentence *Every student is a vegetarian and a shogi player*. Since this sentence is equivalent to the conjunction of two sentences: *Every student is a vegetarian* and *Every student is a shogi player* one can consider, just as an illustration, ignoring obvious syntactic difficulties with such a proposal, that we have here a "complex determiner" which denotes the quantifier *EVERY...IS...AND....* This quantifier is a type $\langle 1, \langle 1, 1 \rangle$ co-cardinal quantifier. Similarly the quantifier *MOST - BUT - NOT - ALL...ARE...AND...* is a generalised cardinal quantifier.

Higher type quantifiers, in addition to their Boolean complements, also have post-complements. A general definition of the post-complement of type $\langle 1^k, 1^l \rangle$ quantifier goes as follows:

- D 24: If F is a type $\langle 1^k, 1^l \rangle$ quantifier then $F\neg$, the post-complement of F , is that type $\langle 1^k, 1^l \rangle$ quantifier for which $F\neg(X_1, \dots, X_k)(Y_1, \dots, Y_l) = F(X_1, \dots, X_k)(Y'_1, \dots, Y'_l)$

The operation of post-complementation has similar effects in the case of higher type quantifiers, as in the case of simple quantifiers. For instance:

- Fact 17: $D \in INT_{\langle 1^n, 1 \rangle}$ iff $D\neg \in CO-INT_{\langle 1^n, 1 \rangle}$ and $D \in INT_{\langle 1, 1^n \rangle}$ iff $D\neg \in CO-INT_{\langle 1, 1^n \rangle}$

Similar conditions hold for cardinal and co-cardinal quantifiers.

It should be clear now how to define symmetric and contrapositional quantifiers of higher types: we have to generalise properties indicated

in propositions 2 and 4. Thus for type $\langle 1^n, 1 \rangle$ quantifiers we have:

D 25: A type $\langle 1^n, 1 \rangle$ quantifier D is symmetric iff there exists a binary commutative function \otimes on pairs of sets such that $\forall X_i, Y_i, Z_1, Z_2$, $D(X_1, \dots, X_n)(Z_1) = D(Y_1, \dots, Y_n)(Z_2)$ if $X_i \otimes Z_1 = Y_i \otimes Z_2$, for every $1 \leq i \leq n$.

D 26: A type $\langle 1^n, 1 \rangle$ quantifier D is contrapositional iff there exists a binary commutative function \otimes such that $\forall X_i, Y_i, Z_1, Z_2$, if $X_i \otimes Z'_1 = Y_i \otimes Z'_2$ then $D(X_1, \dots, X_n)(Z_1) = D(Y_1, \dots, Y_n)(Z_2)$, for every $1 \leq i \leq n$.

Similarly for type $\langle 1, 1^n \rangle$ quantifiers:

D 27: A type $\langle 1, 1^n \rangle$ quantifier is symmetric iff there exists a binary commutative function \otimes on pairs of sets such that for all X_1, X_2, Y_i, Z_i , if $X_1 \otimes Y_i = X_2 \otimes Z_i$, then $D(X_1)(Y_1, \dots, Y_n) = D(X_2)(Z_1, \dots, Z_n)$, for every $1 \leq i \leq n$.

D 28 A type $\langle 1, 1^n \rangle$ quantifier is contrapositional iff there exists a binary commutative function \otimes on pairs of sets such that for all X_1, X_2, Y_i, Z_i , $D(X_1)(Y_1, \dots, Y_n) = D(X_2)(Z_1, \dots, Z_n)$, whenever $X_1 \otimes Y'_i = X_2 \otimes Z'_i$, for every $1 \leq i \leq n$.

The structure of these definitions is the same as the structure of many other definitions proposed above, in particular of intersectivity and co-intersectivity. It allows us to understand in what sense definitions D 25, D 26, D27 and D 28 are generalisations of D 9 and D 10. By generalising the proofs of proposition 2 and of proposition 4 one shows that definition D25 entails in particular a "symmetry-like" property indicated in proposition 9, and D 26 entails a "contrapositional" property given in proposition 10:

Proposition 9: Let $F \in PDET_{\langle 1^n, 1 \rangle}$ and $G \in PDET_{\langle 1, 1^n \rangle}$ such that $F(X_1, \dots, X_n)(Y) = G(Y)(X_1, \dots, X_n)$. Then F is symmetric iff G is symmetric.

Proposition 10: Let $F \in PDET_{\langle 1^n, 1 \rangle}$ and $G \in PDET_{\langle 1, 1^n \rangle}$ such that $F(X_1, \dots, X_n)(Y) = G(Y')(X'_1, \dots, X'_n)$. Then F is contrapositional iff G is contrapositional.

Thus, roughly, proposition 9 says that if two functions have "symmetric types" and are equal then they are both symmetric. We observe also, as for simple quantifiers, that symmetric and contrapositional quantifiers are related by post-complements:

Fact 18: A higher type quantifier F is symmetric iff $F\neg$ is contrapositional.

Furthermore we can also easily establish the following:

Fact 19: If $F \in INT_{\langle 1^n, 1 \rangle}$ or $F \in INT_{\langle 1, 1^n \rangle}$ then F is symmetric.

Fact 20: if $F \in CO-INT_{\langle 1^n, 1 \rangle}$ or $F \in CO-INT_{\langle 1, 1^n \rangle}$ then F is contrapositional.

We can illustrate proposition 9, and indirectly fact 19 by the following example. We will consider two quantifiers we have already seen. Consider first the type $\langle \langle 1, 1 \rangle, 1 \rangle$ quantifier *MORE...THAN...* It is denoted by the binary determiner *more...than...* as it occurs in *More students than teachers*. Its semantics is given by: $MORE(X_1, X_2)(Y) = 1$ iff $|X_1 \cap Y| > |X_2 \cap Y|$. As the second quantifier consider the comparative quantifier *MORE...ARE...THAN...* This type $\langle 1, \langle 1, 1 \rangle \rangle$ quantifier is the denotation of the complex (structured) determiner found in *More students are vegetarians than (are) Buddhists*. Its semantics is given by $MORE(Y)(X_1, X_2) = 1$ iff $|Y \cap X_1| > |Y \cap X_2|$. It follows from the provided semantics that both quantifiers are intersective (in fact *cardinal*) and thus symmetric. Furthermore, we can check that they have both identical semantics up to the permutation of arguments required by the symmetry. So in particular *More students than teachers are Buddhists* is equivalent to *More Buddhists are students than teachers*.

To illustrate proposition 10 we can use the example of higher type (reducible) quantifiers formed by the quantifier *EVERY* and the conjunction *AND*. Thus *Every student and teacher is a Buddhist* is equivalent, under the distributive reading, to *Every non Buddhist is a non-student and a non-teacher*. This means that the (reducible) type $\langle \langle 1, 1 \rangle, 1 \rangle$ quantifier *EVERY...AND...IS...* and the (reducible) type $\langle 1, \langle 1, 1 \rangle \rangle$ quantifier *EVERY...IS...AND...* are both contrapositional.

5. Conclusion

Given the importance of symmetry in cognition and of contraposition in logic we have analysed in parallel two classes of quantifiers: symmetric and contrapositional ones. As a result many new inferential patterns emerged. It appears that the property of (generalised) contraposition is tightly related to symmetry. In fact it can be shown that for any fixed n the algebras of symmetric type $\langle 1^n, 1 \rangle$ and type $\langle 1, 1^n \rangle$ quantifiers are isomorphic to the corresponding algebras of contrapositional quan-

tifiers. Moreover symmetric and contrapositional type $\langle 1, 1 \rangle$ quantifiers are important because any type $\langle 1, 1 \rangle$ quantifier is a Boolean combination of symmetric and contrapositional quantifiers. Finally, it has been shown that the relation between the contrapositional quantifiers and co-intersective ones is similar to that between symmetric quantifiers and intersective ones.

The analysis presented here concerns simple type $\langle 1, 1 \rangle$ quantifiers as well as some higher type quantifiers. The higher type quantifiers we analysed are not considered as just $n+1$ -ary relations between sets (or sets of n -tuples of sets) but as binary relations between n -ary relations and sets. We show how it is possible to generalise symmetry and contraposition to such structured higher type quantifiers even if they relate two objects of different types.

There does not seem to be any formal difficulty in generalising the proposed analysis to all higher type quantifiers which correspond to binary relations over n -ary relations, that is quantifiers of type $\langle 1^k, 1^l \rangle$, for $k > 1$ and $l > 1$. This generalisation was not discussed here because its empirical content is not clear.

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