Set partitions and the meaning of *the same*

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Abstract

It is shown that the notion of the partition of a set can be used to describe in a uniform way the meaning of the expression *the same*, in its basic uses in transitive and ditransitive sentences. Some formal properties of the function denoted by *the same*, which follow from such a description are indicated. These properties indicate similarities and differences between functions denoted by *the same* and generalised quantifiers

1 Introduction

The extensive study of natural language quantifiers with strong application of formal tools in recent years has led to empirical widening of research in that domain. Consequently the semantics and logical properties of various "logically oriented" lexical items, which technically do not denote quantifiers, have also been studied. Among formatives studied on such occasions are *the same* and *different*. These items are of considerable logical interest not only because they are related to quantifiers but because their semantics immediately goes beyond standard first order analysis. In addition, they give rise to specific less-known inferential patterns.

In this paper I will use the notion of a set partition and some notions from (extended) generalised quantifier theory, to provide in particular a more complete semantic description of sentences with *the same* (possibly modified) in some of their basic uses. The semantic description of such constructions is not, however, the main purpose of this paper. I am more interested in the description of the type of functions denoted by *the same CN* and in their formal properties. Thus I will indicate some formal properties of functions denoted by these items, properties which will specify what these functions have in common with (generalised) quantifiers in particular. In addition, I will indicate some inference patterns based on them.

The items that will be considered in this paper have many uses which have been studied in numerous publications (see Barker 2007 for an overview of an important part of the relevant literature on the subject). My main objection to most descriptions of sentences with *the same* concerns their lack of generality: they do not account for the fact that subject NPs of such sentences are in general not constrained, except for plurality. Thus subject NPs of sentences with *the same CN* in the object position can denote not only (principal) filters but quantifiers of various other types including those formed with numerals, proportional quantifiers or even quantifiers taking many set arguments. For instance the following NPs can easily occur in subject position in sentences with *the same*: *ten teachers, no student, except Leo and Lea, most students, including Leo and Lea, at least seven monks, most but less than seventeen female students, more philosophers than logicians, between seven and eleven teachers*, etc.

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Given that fact that *the same* has many uses and can be categorially characterised in multiple ways, it is useful to indicate the uses of *the same* that will not be studied here.

First, the expression *the same* has been studied in the context of binary nominal determiners which denote type $\langle 1, 1, 1\rangle$ quantifiers, that is functions taking 2 sets as arguments and giving a type $\langle 1\rangle$ quantifier as results (Beghelli 1994, Keenan and Moss 1985, Zuber 2009). The following examples illustrate such a use of *the same*:

(1) The same students came early as left late.

Second, we will not study *the same* occurring in comparative constructions like (2):

(2) Leo read the same book as most students.

Observe that strictly speaking in (2) we have not *the same* but rather *the same as*. This use of *the same as* has been partially studied in Zuber (2011).

Finally, I will be interested in the sentence internal reading of this expression and not in its sentence external reading. Roughly, this means that the "antecedent" of *the same CN* has to be in the same sentence. Consider the sentences in (3) in comparison with those in (4):

(3a) Leo and Lea read the same books.
(3b) Leo read the same books as Lea and Lea read the same books as Leo.
(4a) Dan read *Exciting Humour* and *Hot Logic*. Leo and Lea read the same books.
(4b) Dan read *Exciting Humour* and *Hot Logic*. Leo read the same books.

The sentence external reading of *the same book* is most naturally obtained in (4a) and (4b) since *the same book* somehow is related to an element not present in the second sentence in the sequence of two sentences. In (3a), even though such an external reading is possible, we get easily sentence internal reading where (3a) is equivalent to (3b). Furthermore, in sentence external reading the "sameness" is only partial. For instance (4a) does not entail that all the books that Leo and Lea read are the same. This is not the case in (3a) since (5) sounds contradictory:

(5) Leo and Lea read the same books and in addition Leo read *Hot logic*

Observe also that sentence external reading, in opposition to internal reading, is possible with singular subjects and with intransitive sentences:

(6) Leo read *Exciting Humour*. Lea read the same book.
(7) Two students danced. The same students sang (as well).

Finally, it seems that external *the same* is "redundant" in the sense that it can be replaced by "logically simpler" items. For instance (4a) and (7) are equivalent to (8) and (9) respectively:

(8) Dan read *Exciting Humour* and *Hot Logic*. Leo and Lea read them too.
(9) Two students danced. They also sang.

The second point concerning the data to be considered has, roughly speaking, to do with what philosophers of language would call the *type-token* distinction. Obviously *the same* involves some kind of identity between objects and thus when interpreting sentences with *the same* a question arises as to just how similar two objects have to be in order to count as the same. In (3a) for instance both identities, type identity and token identity, are possible: Leo and Lea could have read the same token of the book or the same "type", roughly speaking, including even the possibility of being just a translation of some book. Observe that this example shows that type-token distinction is not really appropriate, and probably, as shown in Lasersohn (2000) the degree of similarity matters. I will ignore possible complications due to this distinction.

Concerning the syntax of the sentences to be analysed I will assume a simple categorial grammar with mostly classical major categories such as NP- a noun phrase, CN - a common noun, a VP a verb phrase and TVP - a transitive verb phrase. CN are supposed to denote sets, subsets of a given universe, TVP denote binary relations and $DTVP$- ditransitive verb phrases, denote ternary relations over this universe. In this framework the expressions *the same* and *the same number of* will be considered as (generalised) determiners: they take a CN as arguments and give a nominal argument of $TVPs$ or of $DTVPs$ as result.

To recapitulate the above remarks we will consider sentences having one of the following forms:

(10a) $NP\ TVP\ THE\ SAME\ CN$
(10b) $THE\ SAME\ TVP\ NP$
(10c) $NP_1\ DTVP\ NP_2\ Prep\ THE\ SAME\ CN$
(10d) $NP\ DTVP\ THE\ SAME\ CN_1\ Prep\ THE\ SAME\ CN_2$
(10e) $NP\ TVP_1\ AND\ TVP_2\ THE\ SAME\ CN$

As indicated in (10d) we will also analyse sentences in which two $THE\ SAME$ CN occur: one in the direct object position and one in the indirect object position as illustrated in (11):

(11) Leo and Lea gave the same books to the same children.

In addition we will analyse in similar sentences a modified *the same*, that is the determiner *the same number of*. However, only sentences of the form (10a) and (10e) will be analysed in some details. The analysis of sentences of the forms (10b), (10c) and (10d) will essentially use the analysis of sentences of the form (10a).

As the forms in (10) show we are interested in *the same CN* playing the role of verbal arguments as "ordinary" NPs. It is well-known, however, that there are also non-verbal "transitive expressions" which can take NPs as arguments and form (generalised) predicatives. For instance, there are transitive CNs such as *grand-parents of* and transitive adjectival phrases such as *jealous of*, which form "simple" CNs or "simple" adjectival phrases with NPs. Interestingly, *the same CN* can also occur as an argument in such constructions since we have *grand-parents of the same students* and *jealous of the same composers.* Probably other expressions with embedded *the same CN* such as *articles with the same errors* or *books with the same number of chapters* are of the same type. Although these constructions will not be analysed here, it should be stressed that their semantics is likely to involve binary relations.

What is the categorial status of *the same* and the type of function it denotes? We know that sentences with *the same*, of the form given in (10a) do not take proper nouns as subjects (under the sentence internal readings) and thus the type of objects denoted by the subject NP cannot be e, the type corresponding to individuals. We can suppose that it is of the raised type $\langle\langle e,t\rangle,t\rangle$, which, ignoring directionality, corresponds to the category $S/(S/NP)$. Since *the same* applied to a common noun forms a verbal argument playing the role of direct object *the same (CN)* applies to a transitive verb to form a VP. We will consider, that semantically this VP denotes a set of type $\langle 1\rangle$ quantifiers. Thus, in order to avoid the type mismatch the verb phrase must be raised to become of the category $S/(S/(S/NP))$. This move accounts naturally for the fact that almost any NP can occur as subject NP in "transitive" sentences with *the same (CN)* in the object position.

A semantic observation going in this direction concerns the behaviour of conjoined NPs in the subject position. Consider (12) and (13):

(12a) Leo and Lea read the same books.
(12b) Bill and Sue read the same books.
(13) Leo, Lea, Bill and Sue read the same books.

Clearly (12a) in conjunction with (12b) does not entail (13). This means that the functions denoted by the subject NPs in (12a) and (12b) do not apply to the predicate denoted by the complex VPs in these sentences and thus the conjunction *and* is not understood pointwise. Hence, to avoid the type mismatch and get the right interpretations we will consider that in the basic case of sentences like (3a) the verbal argument *the same books* denote higher order functions that is functions taking a binary relation as argument and giving a set of quantifiers as output. This analysis will be assumed in what follows.

2 Formal Preliminaries

Let me start by recalling some basic notions from the "classical" generalised quantifier theory and some less classical extensions and generalisations of them which will be used in what follows. As we will see these extensions have a basically linguistic justification.

Given a fixed universe E, (where $|E| \geq 2$), a type n quantifier is a function from n-ary relations to truth values. A type $\langle 1\rangle$ quantifier is a function from sets (sub-sets of E) to truth values, and thus it is a set of sub-sets of E. A type $\langle 1,1\rangle$ quantifier is a function from sets to type $\langle 1\rangle$ quantifiers. In natural language semantics type $\langle 1\rangle$ quantifiers are denotations of NPs and a type $\langle 1,1\rangle$ quantifiers are denotations of (unary nominal) determiners, that is expressions like *every, no, most, five*, etc. Since both types of quantifiers form Boolean algebras they have Boolean complements (negations): If Q is a quantifier (of one of the above indicated types) then $\neg Q$ is its Boolean complement. Both type of quantifiers have also *post-negations*: if Q is a type $\langle 1\rangle$ quantifier then its post-negation $Q\neg$ is defined as: $Q\neg = \{X : X' \in Q\}$, where X' is the Boolean complement of X. If Q is a type $\langle 1,1\rangle$ quantifier then $Q\neg$ is that type $\langle 1,1\rangle$ quantifier which for every set X associates to $Q(X)$ the post-complement of $Q(X)$: $Q\neg(X) = Q(X)\neg$. These two types of negation allow us to define the dual quantifier Q^d of a given quantifier Q: $Q^d = \neg(Q\neg) = (\neg Q)\neg$. For instance $EVERY$ and $SOME$ (considered as type $\langle 1,1\rangle$) are dual of each other. Similarly, the quantifier $EVERY$-BUT-AT-$MOST$ n is the dual of AT-$LEAST$ n.

A type $\langle 1\rangle$ quantifier is atomic iff it contains only one element. In our notation Q_A denotes the atomic quantifier which has just A as its unique element.

As the examples presented above show, sentences with *the same CN* necessitate plural subjects and thus we need the notion of a plural type $\langle 1\rangle$ quantifier, denotation of a plural NP. A fully satisfactory definition of the notion of a plural quantifier may appear complicated. For our purposes we will use the following definition: the quantifier Q is called *plural*, in symbols $Q \in PLR$, iff $Q \subseteq 2(E)$ or $\neg Q \subseteq 2(E)$, where $2(E) = \{X : X \subseteq E \wedge |X| \geq 2\}$.

A type $\langle 1 \rangle$ quantifier is monotone increasing, $Q \in MON$, iff $Q(Y_1) \subseteq Q(Y_2)$ whenever $Y_1 \subseteq Y_2$. Some (type $\langle 1 \rangle$) quantifiers are filters generated by a set, or principal filters: $Ft(A)$ is a filter generated by A iff $Ft(A) = \{X : A \subseteq X\}$.

The above notions belong to the "classical" generalised quantifier theory. This means that basically type $\langle 1 \rangle$ quantifiers are functions from sets to truth values. Linguistically such functions interpret noun NPs in (grammatical) subject positions. However, NPs can occur not only in subject position and thus their domain of applications should be extended so that they also apply n-ary relations. Here we will consider only a particular case of extension to binary and ternary relations (supposed to be denotations of TVPs and of $DTVP$s respectively).

A type $\langle 1 \rangle$ quantifier Q can apply in two ways to binary relations: as a "subject-function", called its nominative extension, Q_{nom} and as a "direct object function" called the accusative extension, Q_{acc}. They are formally defined in the following way. Let Q be a type $\langle 1 \rangle$ quantifier, R a binary relation and $a \in E$. Then $aR = \{x : \langle a, x \rangle \in R\}$ and $Ra = \{x : \langle x, a \rangle \in R\}$. Given this notation Q_{nom} and Q_{acc} are defined as follows:

D1: $Q_{nom}(R) = \{x : Q(Rx) = 1\}$
D2: $Q_{acc}(R) = \{x : Q(xR) = 1\}$.

If it is clear that R is a binary relation $Q_{nom}(R)$ will be noted $Q(R)$ from now on.

Case extensions of a type $\langle 1 \rangle$ quantifier can be used to make clear the notions of subject and object wide scope readings. Thus given sentence (14) in which the (subject) NP_1 denotes the quantifier Q_1 and the (object) NP_2 denotes Q_2 and TVP denotes the binary relation R, in (15a) we have the subject wide scope reading and in (15b) the object wide scope reading of (14):

(14) NP_1 TVP NP_2
(15a) Q_1 $((Q_2)_{acc}(R))$
(15b) $Q_2((Q_1)_{nom}(R))$

As we will see some readings of *the same CN* are related to wide scope readings.

Definitions D1 and D2 indicate how the domain of a type $\langle 1 \rangle$ quantifier can be extended to include binary relations. The extension defined in D2 shows how to calculate the denotation of the VP obtained by the composition of a TVP with a NP playing the role of the direct object (interpreted with the narrow scope). Since we will also deal with $DTVP$s we also need to extend the domain of type $\langle 1 \rangle$ quantifiers so that they apply to ternary relations. To do this we need the following definitions:

D3: Let S be a ternary relation. Then:
(i) $\langle a_1, a_2 \rangle S = \{x : \langle a_1, a_2, x \rangle \in S\}$
(ii) $\langle a_1, a_3 \rangle S = \{x : \langle a_1, x, a_3 \rangle \in S\}$
(iii)$\langle a_2, a_3 \rangle S = \{x : \langle x, a_2, a_3 \rangle \in S\}$
(iv) $a_1 S = \{\langle y, z \rangle : \langle a_1, y, z \rangle \in S\}$

The application of a type $\langle 1 \rangle$ quantifier to a ternary relation can now be defined as follows:

D4: Let S be a ternary relation and Q a type $\langle 1\rangle$ quantifier. Then:

(i) $Q_{3c}(S) = \{\langle x_1, x_2\rangle : Q(\langle x_1, x_2\rangle S) = 1\}$
(ii) $Q_{2c}(S) = \{\langle x_1, x_3\rangle : Q(\langle x_1, x_3\rangle S) = 1\}$
(ii) $Q_{1c}(S) = \{\langle x_2, x_3\rangle : Q(\langle x_2, x_3\rangle S) = 1\}$

The definitions in D4 indicate how a type $\langle 1\rangle$ quantifier reduces a ternary relation to a binary one. Such a reduction happens when, syntactically speaking, an NP becomes an argument of a $DTVP$. The position of the argument is indicated by the indices of coordinates. For instance Q_{3c} corresponds to the situation in which the argument NP is the third argument of the $DTVP$.

We will consider binary and ternary relations and functions over the finite universe E. If a function takes only a binary relation as argument, its type is noted $\langle 2 : \tau\rangle$, where τ is the type of the output; if a function takes a set and a binary relation as arguments, its type is noted $\langle 1, 2 : \tau\rangle$. If $\tau = 1$ then the output of the function is a set of individuals and thus the type of the function is $\langle 2 : 1\rangle$. For instance the function $SELF$, where $SELF(R) = \{x : \langle x, x\rangle \in R\}$, is of this type. Case extensions exemplify this type of function as well.

The case we will basically consider here is when τ corresponds to a set of type $\langle 1\rangle$ quantifiers and thus τ equals, in Montagovian notation, $\langle\langle\langle e, t\rangle t\rangle t\rangle$. In short, the type of such functions will be noted either $\langle 2 : \langle 1\rangle\rangle$ (functions from binary relations to sets of type $\langle 1\rangle$ quantifiers)) or $\langle 1, 2 : \langle 1\rangle\rangle$ (functions from sets and binary relations to sets of type $\langle 1\rangle$ quantifiers). Similarly, the type of functions mapping ternary relations to binary relations is noted $\langle 3 : 2\rangle$ and the type $\langle 1, 3 : 2\rangle$ corresponds to functions having sets and ternary relations as arguments and binary relations as results. Definition D4 provides an example of such functions. Finally, the type of functions mapping ternary relations to sets of binary relations is noted $\langle 3 : \langle 2\rangle\rangle$ and the type of functions mapping sets and ternary relations to sets of binary relations will be noted $\langle 1, 3 : \langle 2\rangle\rangle$. Such functions will be called *higher order* functions (on relations). As we will see, *the same (CN)* denotes a higher order function.

If R is a binary relation and A a set then R_A denotes a sub-relation of R whose range is restricted by A that is $R_A = \{\langle x, y\rangle : \langle x, y\rangle \in R \wedge y \in A\}$. Similarly if S is a ternary relation $S_{A,B}$ denotes a sub-relation of S whose second elements of triples that belong to S are restricted to A, and third elements of these triples are restricted to B, that is $S_{A,B} = \{\langle x, y, z\rangle : \langle x, y, z\rangle \in S \wedge y \in A \wedge z \in B\}$.

Any mapping f of a given set A (on another set) establishes an equivalence relation e_f on A defined as $\langle x, y\rangle \in e_f$ iff $f(x) = f(y)$. Let R be a binary relation. It gives rise to the following two mappings of E: (1) $f_1(x) = xR$ and (2) $f_2(x) = |xR|$. This means that with any binary relation R one can associate the following two equivalence relations:

D5 (i) $e_R = \{\langle x, y\rangle : xR = yR\}$
(ii) $e_{R,n} = \{\langle x, y\rangle : |xR| = |yR|\}$

Similarly, with any ternary relation S we can associate the following equivalence relations:

D6 (i) $e_{2cS} = \{\langle\langle a_1, a_3\rangle, \langle b_1, b_3\rangle\rangle : \langle a_1, a_3\rangle S = \langle b_1, b_3\rangle S\}$
(ii) $e_{2cS,n} = \{\langle\langle a_1, a_3\rangle, \langle b_1, b_3\rangle\rangle : |\langle a_1, a_3\rangle S| = |\langle b_1, b_3\rangle S\}|$
(iii) $e_S = \{\langle a, b\rangle : aS = bS\}$

We will consider various partitions of a set corresponding to the above equivalence relations. If R is a binary relation then $\Pi_R(X)$ is the partition of the set X defined by the relation e_R and $\Pi_{R,n}(X)$ is the partition of X defined by the relation $e_{R,n}$. Similarly, if S is a ternary relation then $\Pi_{2cS}(X)$ is the partition of the set X of ordered pairs defiined by the relation e_{2cS} and $\Pi_{2cS,n}(X)$ is the partion of the set X of ordered pairs defined by the relation $e_{2cS,n}$. Finally D6 (iii) defines an equivalence relation induced by the ternary relation S to which corresponds the partition $\Pi_S(X)$ induced by the relation S.

Since a partition (of a set of individuals) is a set of sets (of individuals) one can consider that D5 associates with any binary relation R a set of unary relations and D6 associates with any ternary relation S a set of binary relations - a partition of ordered pairs.

Partitions (of the same set) can be partially ordered by the refinement relation: the partition $\Pi_1(X)$ refines the partition $\Pi_2(X)$ iff for any block $B_1 \in \Pi_1(X)$ there exists a block $B_2 \in \Pi_2(X)$ such that $B_1 \subseteq B_2$. It is easy to see that $\Pi_R(X)$ refines $\Pi_{R,n}(X)$. Moreover we also obtain a refinement relation on some of the above partitions, if we consider relations R_C and R_D with C and D ordered by inclusion. More specifically we have:

Proposition 1: Let $C \subseteq D$. Then for any $X \neq \emptyset$, any $R \neq \emptyset$, any $S \neq \emptyset$: $\Pi_{R_D}(X)$ refines $\Pi_{R_C}(X)$

Given that $aR = bR$ iff $aR' = bR'$, $aR = aS$ iff $aR' = aS'$ and $|aR| = |aS|$ iff $|aR'| = |aS'|$, for R and S binary, we also have:

Proposition 2: (i) $\Pi_R(X) = \Pi_{R'}(X)$
(ii) $\Pi_{R,n}(X) = \Pi_{R',n}(X)$

Quantifiers found in natural language obey various constraints which can be used to distinguish various classes of quantifiers (conservative, intersective, cardinal, etc., see Keenan and Westerståhl 1997). For instance, a type $\langle 1,1\rangle$ quantifier D is conservative iff $D(X,Y) = D(X, X \cap Y)$. Similarly, D is intersective iff $D(X,Y) = D(X \cap Y, X \cap Y)$ and D is co-intersective iff $D(X,Y) = F(X - Y, X' \cup Y)$. Intersective and co-intersective quantifiers form atomic Boolean algebras INT and $CO\text{-}INT$ respectively. Their atoms are determined by sets. Thus the type $\langle 1,1\rangle$ quantifiers D_A (for $A \subseteq E$) is an atom of INT iff $D_A(X,Y) = 1$ iff $X \cap Y = A$. Similarly, D_A is an atom of $CO\text{-}INT$ iff $D_A(X,Y) = 1$ iff $X \cap Y' = A$. For instance the determiner *no...except Leo and Lea* denotes the atom of INT determined by the set containing exactly Leo and Lea.

The algebra INT has a sub-algebra $CARD$ and the algebra $CO\text{-}INT$ has a sub-algebra $CO\text{-}CARD$. By definition (Keenan and Westerståhl 1997) $D \in CARD$ (resp. $D \in CO\text{-}CARD$) iff $D(X_1,Y_1) = D(X_2,Y_2)$ whenever $|X_1 \cap Y_1| = |X_2 \cap Y_2|$ (resp. $|X_1 \cap Y_1'| = |X_2 \cap Y_2'|$. Both classes of quantifiers form atomic Boolean algebras whose atoms are determined by cardinals. Functions $EXACT_n$ such that $EXACT_n(X,Y) = 1$ iff $|X \cap Y| = n$ are atoms of $CARD$ and functions $EXACT_n$ such that $EXACT_n(X,Y) = 1$ iff $|X \cap Y'| = n$ are atoms of $CO\text{-}CARD$ (for $n \in N$). For instance the determiner *every... but ten* denotes the atom of $CO\text{-}CARD$ determined by the cardinal 10.

All these properties of quantifiers can be generalised so that they apply to simple and higher order functions. The following definitions will be used (cf. Zuber 2010):

D7: A function F of type $\langle 1, 2 : \tau \rangle$ is conservative iff $F(X, R) = F(X, (E \times X) \cap R)$
D8: A type $\langle 1, 2 : \tau \rangle$ function F is intersective (resp. co-intersective) iff $F(X_1, R_1) = F(X_2, R_2)$ whenever $(E \times X_1) \cap R_1 = (E \times X_2) \cap R_2$ (resp. $(E \times X_1) \cap R'_1 = (E \times X_2) \cap R'_2$).
D9: A type $\langle 1, 2 : \tau \rangle$ function is cardinal (resp. co-cardinal) iff $F(X_1, R_1) = F(X_2, R_2)$ whenever $\forall y(|X_1 \cap yR_1| = |X_2 \cap yR_2|)$ (resp. $\forall y(|X_1 \cap yR'_1| = |X_2 \cap yR'_2|)$).

One can notice that cardinal functions are intersective and intersective functions are conservative. Similarly, co-cardinal functions are co-intersective which, themselves, are conservative.

We will also analyse sentences with *the same* in which two transitive verbs occur. In this case functions taking two relational arguments (that is functions of type $\langle 1, 2^2 : \langle 1 \rangle \rangle$ are involved. For them we can also define (generalised) conservativity, intersectivity, etc. Thus we have (for R_1, R_2, S_1, S_2 binary):

D10: A type $\langle 1, 2, 2 : \tau \rangle$ function F is conservative iff if $(E \times X) \cap R_1 = (E \times X) \cap R_2$ and $(E \times X) \cap S_1 = (E \times X) \cap S_2$ then $F(X, R_1, S_1) = F(X, R_2, S_2)$
D11: A type $\langle 1, 2, 2 : \tau \rangle$ function F is intersective iff if $(E \times X_1) \cap R_1 = (E \times X_2) \cap R_2$ and $(E \times X_1) \cap S_1 = (E \times X_2) \cap S_2$ then $F(X_1, R_1, S_1) = F(X_2, R_2, S_2)$
D12: A type $\langle 1, 2 : \tau \rangle$ function F is cardinal (resp. co-cardinal) iff $F(X_1, R_1, S_1) = F(X_2, R_2, S_2)$ whenever $\forall y(|X_1 \cap yR_1| = |X_2 \cap yR_2|)$ and $\forall y(|X_1 \cap yS_1| = |X_2 \cap yS_2|)$ (resp. $\forall y(|X_1 \cap yR'_1| = |X_2 \cap yR'_2|)$ and $\forall y(|X_1 \cap yS'_1| = |X_2 \cap yS'_2|)$).

As in the case for type $\langle 1, 1 \rangle$ quantifiers it is possible to give other, equivalent, definitions of conservativity, intersectivity and co-intersectivity of type $\langle 1, 2 : \tau \rangle$ functions. I will use the definition of intersectivity given by:

Proposition 3: A type $\langle 1, 2 : \tau \rangle$ function F is intersective (resp. co-intersective) iff $F(X, R) = F(E, (E \times X) \cap R)$ (resp. $F(X, R) = F(E, (E \times X') \cup R)$) .
Proposition 4: A type $\langle 1, 2^2 : \tau \rangle$ function F is intersective iff $F(X, R, S) = F(E, (E \times X) \cap R, (E \times X) \cap S)$.

Observe that most of the above definitions do not depend on the type τ of the result of the application of the function. So obviously they can be used with higher order functions.

3 Application of partitions

Before showing how the notion of a set partition applies to the analysis of sentences with *the same* some remarks about readings of such sentences are in order.

I will be essentially interested in the *logical*, that is not involving existential import, semantic interpretation of constructions with *the same*. On this interpretation of (16), for instance, it is neither presupposed nor entailed that any reading of any book by any student took place and thus (16) does not entail (17) neither on the subject not on the object wide scope readings of (16):

.
(16) Every student read the same books.
(17) Every student read some book(s).

Thus on this "logical" reading (16) is true if no student read any book. Observe that (18a) and (18b) do not sound as contradictory and consequently such a reading is not absurd:

(18a) Every student read the same book, namely none.
(18b) Every student answered the same question, namely none.

The restriction to readings without existential import is not essential. It will be shown below how we can obtain the semantics of *the same* which takes into account the existential import as well.

There might be differences between the case where *the same* applies to a singular CN and the case when it applies to a plural CN. For instance (19a) has probably a reading with a "partial sameness" which does not entail that all students read exactly one book. This reading can be expressed by the object wide scope reading of the quantifier $EXACTLY\text{-}ONE$ or even $SOME$, as in (19b) or (19c):

(19a) Every student read the same book.
(19b) $EXACTLY\text{-}ONE(BOOK)(EVERY(S)_{nom})(READ)$
(19c) $SOME(BOOK)(EVERY(S)_{nom})(READ)$

Such a reading with scope inversion clearly occurs with the expression like *the same n CN* as illustrated in (20). In this sentence the set of books read by every student needs not to be the same as shown by the non-contradiction of (21):

(20) Every student read the same five books.
(21) Every student read the same five books and ten different.

It seems that in sentences with *the same CN*, where CN is in plural, the reading equivalent to the reading with inversed scope, that is the reading with "partial sameness" is not possible as witnessed by the contradiction of (22):

(22) Leo and Lea read the same books and Leo, but not Lea, read in addition *Exciting Humor*

What (16) essentially says is that no book is such that it was read by some but not by all students or, equivalently, that any book that was read by any student was read by every student. Thus (16) should satisfy condition given in (23a) or (23b):

(23a) $NO(B)[(SOME(S) \wedge \neg EVERY(S)]R = 1$
(23b) $SOME(S)R \cap B \subseteq EVERY(S)R \cap B$

Sentence in (16) is just a particular case of constructions with *the same* occurring in the direct object NP. An important point is that in (16) the quantified NP in the subject position can be replaced by many other quantified NPs, and even non-quantified ones. This can be seen in (24):

(24) No students/most students/five students/at least two students/Leo and Lea read the same book(s)

It is not easy to describe restrictions on noun phrases in sentences of the above type. It seems clear that semantically they should denote plural quantifiers. For instance the interpretation of (16) should be the same as the interpretation of (25a) and (25b) should have the same interpretation as (25c):

(25a) Every (group of) two students read the same books.
(25b) No students read the same books.
(25c) No two students read the same books.

In order to establish the relationship between set partitions and the semantics of *the same*, we start with the observation that any mapping of A establishes an equivalence relation on A. In the case at hand, we can take, informally, the mapping which associates with $x \in E$ the set of "the same X" modulo the relation R. More precisely, the function $SAME(X, R)$ partitions the universe E in the way that any block B of the partition is related to a subset X_1 of X by the fact that any member of B is in a relation R with any member of X_1. The quantifier corresponding to the subject NP can be true or not of such a block of the partition.

We can thus specify the function $SAME(X, R)$, denoted by the determiner *the same* using the partition $\Pi_{R_X}(E)$. The definition to be given will be definition "by cases". The output of the function to be defined, that is a set of plural type $\langle 1 \rangle$ quantifiers, will in general contain three parts: positive, negative and "atomic". The positive part corresponds, roughly, to the set of quantifiers true of some block of the partition and the negative part corresponds to the set of quantifiers false of sets which are not blocks of the partition.

We will say that the block of a partition is singular if it is a singleton. A block B is plural, $B \in PL$, if it is contains at least two elements. A partition is atomic iff all its blocks are singular. With the help of these notions, using the partition $\Pi_{R_A}(E)$ we can now express the function $SAME(X, R)$, where R is a binary relation, as follows (for X and R non-empty):

D13 $SAME(X, R) =$
(i)=$\{Q : Q \in PLR \wedge \neg 2(E) \subseteq Q\}$, if $\Pi_{R_X}(E)$ is atomic
(ii)= $\{Q : Q \in PLR \wedge \exists_B (B \in \Pi_{R_X}(E) \wedge B \in PL \wedge Q(B) = 1)\} \cup$
$\{Q : Q \in PLR \wedge \exists_{C \subseteq E} \forall_{B \in \Pi_{R_X}(E)} (C \not\subseteq B \wedge \neg ALL(C) \subseteq Q)\}$, if $\Pi_{R_X}(E)$ is not atomic.

The above definition says that $SAME$ applied to a set X and a binary relation R gives as result a set of quantifiers. This set can be decomposed into various sub-sets depending on the structure of the partition of E induced by R and X. Clause (i) says that when the partition is atomic then no two objects are in the relation R with all objects of a sub-set of X. This entails that the quantifier denoted by *no two objects* and any of its consequences belong to the set $SAME(X, R)$. This means that, for instance, the quantifiers denoted by *no five objects* or *no two students* also belong to the set $SAME(X, R)$.

Clause (ii) concerns the case where the partition is not atomic. In this case there is at least one plural block of the partition such that all its members are, roughly speaking, in the relation R with the same subset of X. This block corresponds to the property expressing the sameness we are looking for and a plural quantifier can be true or false of it. The second part of the clause (ii) provides a set of quantifiers obtained from a "negative information" given by sets which are not blocks of the partition. If, for instance, Jiro and Taro are Japanese students who read different books then no set to which they belong is a block of $\Pi_{R_B}(E)$ - where R corresponds to $READ$ and B - to $BOOK$. Then, according to the second part of the clause (ii), the quantifiers denoted by the NPs *not all Japanese students*, *not all students* and *not all Japanese* belong to $SAME(B, R)$.

In D13 we use explicitly Boolean complements of some quantifiers. Interestingly, Boolean complements of quantifiers can frequently occur in sentences with *the same*. Thus probably (26) is a sloppy way of expressing (27) where the Boolean complement of a type $\langle 1 \rangle$ quantifier explicitly occurs:

(26) No students read the same book.
(27) No two students read the same book.

Consider now example (28a). I take it for granted that (32a) is equivalent to (28b) and thus that it is true if and only if (29) is false:

(28a) No three students read the same books.
(28b) It is not true that three students read the same books.
(29) Three students read the same books.

In (28a) the NP *no three students* denotes the quantifier $\neg 3(S)$, where $3(S)(Y) = 1$ iff $|S \cap Y| \geq 3$. Thus if there is no bloc of which $3(S)$ is true then the quantifier $\neg 3(S)$ belongs to $SAME(B, R)$.

Natural languages also have verbal arguments containing *the same* which can express "sameness of cardinality" as in the following examples:

(30) Leo and Lea read the same number of books.
(31) Most Japanese know the same number of languages.

In these examples *the same number of* is a higher order determiner which forms a veral argument when applied to a CN. So it denotes a type $\langle 1, 2 : \langle 1 \rangle \rangle$ function as does the determiner *the same*. It seems natural and obvious to use the partition induced by the equivalence relation $e_{R,n}$ to describe this function. Thus the definition of the function $SAME$-N denoted by the generalised determiner *the same number of* is quite similar to the definition of the function $SAME(X, R)$. We just have to replace everywhere in D13 the partition $\Pi_{R_X}(E)$ by the partition $\Pi_{R_X,n}(E)$. Consequently we have:

D14: $SAME$-$N(X, R)$=
(i)=$\{Q : Q \in PLR \wedge \neg 2(E) \subseteq Q\}$, if $\Pi_{R_{X,n}}(E)$ is atomic
(ii)= $\{Q : Q \in PLR \wedge \exists_B (B \in \Pi_{R_{X,n}}(E) \wedge B \in PL \wedge Q(B) = 1)\} \cup$
$\{Q : Q \in PLR \wedge \exists_{C \subseteq E}(C \notin \Pi_{R_X}(E) \wedge \neg ALL(C) \subseteq Q)\}$, if $\Pi_{R_X}(E)$ is not atomic.

Definitions D13 and D14 provide the readings of *the same* and *the same number of* without the existential import. In order to get the reading in which the existential import is involved the following equivalence relations have to be used:

(32) $e^{ei}_R = \{\langle x, y \rangle : (xR = yR \wedge xR \neq \emptyset) \vee (x = y)\}$
(33) $e^{ei}_{R,n} = \{\langle x, y \rangle : (|xR| = |yR| \wedge xR \neq \emptyset) \vee (x = y)\}$

The relation e^{ei}_R defines the partition $\Pi^{ei}_R(E)$ and the relation $e^{ei}_{R,n}$ defines the partition $\Pi^{ei}_{R,n}(E)$. It follows from (32) and (33) that if $aR = \emptyset$, then the singleton $\{a\}$ is a singular block of both partitions Π^{ei}_R and $\Pi^{ei}_{R,n}$ and thus is not a member of any plural quantifier. Consequently the reading of *the same* with the existential import is given in D15:

D15: $SAME^{ei}(X, R)$=
(i)=$\{Q : Q \in PLR \wedge \neg 2(E) \subseteq Q\}$, if $\Pi^{ei}_{R_X}(E)$ is atomic
(ii)= $\{Q : Q \in PLR \wedge \exists_B (B \in \Pi^{ei}_{R_X}(E) \wedge B \in PL \wedge Q(B) = 1)\} \cup$
$\{Q : Q \in PLR \wedge \exists_{C \subseteq E} \forall_{B \in \Pi^{ei}_{R_X}(E)} (C \not\subseteq B) \wedge \neg ALL(C) \subseteq Q)\}$, if $\Pi^{ei}_{R_X}(E)$ is not atomic.

The expressions *the same CN* and *the same number of CN* can also occur, though less frequently, in subject position:

(34a) The same actors played three characters of the movie.
(34b) The same number of errors occurred in most papers.

In order to express a reading with the the "full sameness" in such sentences we have to take the inverse R^{-1} of the binary relation R denoted by the TV of such sentences. This means that the partitions to be used in the analysis of sentences with *the same CN* are $\Pi_{R^{-1}}(E)$ and $\Pi_{R^{-1},n}(E)$

When describing the semantics of ditransitive sentences with *the same* one has to distinguish various cases, all of which will not be discussed here. The simplest case is when *the same CN* occurs in the indirect object position and its antecedent, the plural NP, occurs in the subject position:

(35) Leo and Lea gave five books to the same children.

Recall (cf. D4) that type $\langle 1 \rangle$ quantifier can be treated as arity reducing function: it applies to a ternary relation and gives a binary relation. One can consider that in the above example the quantifier $FIVE(BOOK)$ applies to the ternary relation $GIVE...TO$ and gives a binary relation $GIVE$-$FIVE(BOOK)$-TO. Then the function $SAME$ applies to this relation in the way indicated in D13 or in D15.

As shown in (11) it is possible to have ditransitive sentences in which *the same CN* occurs twice: in the direct object position and in the indirect object position. The pretheoretical interpretation of such sentences is not easy and their truth conditions are not obvious. However, the interpretation of such sentences also involves partitions of sets. These partitions are induced by a ternary relations. Consider again the example (36):

(36) Leo and Lea gave the same books to the same children.

It might be tempting to analyse such sentences using the equivalence relation defined in D6(iii): we look at the blocks of the partition which contain the first elements of the triples which have the same second elements and the same third elements. This move should be abandoned, however. First, this way of semantic calculation is probably not compositional since it entails that there is a syntactic rule which combines two verbal arguments, both of the form *the same CN*, before one of them applies to the ditransitive verb. Second, the semantic result obtained in this way is not satisfactory. To see this, consider the ternary relation given in (37):

(37) $S = \{\langle a_1, , b_1, c_1 \rangle, \langle a_2, b_1, , c_1 \rangle, \langle a_1, b_2, c_2 \rangle, \langle a_2.b_2, c_2 \rangle, \langle a_3, b_2, c_2 \rangle\}$

Using definition D6 (iii) and the relation S in (37) we obtain as one of the blocks

$\{a_1, a_2\}$. This means that if S corresponds to the relation *GIVE-TO* in (36) then a_1 and a_2 gave the same book (b_1) to the same child (c_1). Informally this is true but in addition it is also true that a_1, a_2 and a_3 gave the same book (b_2) to the same child (c_2). This last information cannot be obtained from the partition defined in D6 (iii) since the set $\{a_1, a_2, a_3\}$ is not a block of the partition..

The way out is to reduce a ternary relation in sentences like (37) to a set of binary relations using an equivalence relation like the one in D6 (i) and then reduce these binary relations to a set of unary relations, that is a set of sets and then proceed, roughly speaking, as indicated in D13.

Before clarifying this idea we need to modify the equivalence relation in definition D6 (ii) in the way the equivalence relations in D5 were modified to get readings with the existential import according to definitions given in (32) and (33). The reason is that to interpret (36) in a purely logical way without supposing that any books were given to any children is very unnatural. Consequently given a ternary relation S we define the following equivalence relations induced by S:

(38a) $e_{2cS} = \{\langle\langle a_1, a_3\rangle, \langle b_1, b_3\rangle\rangle : (\langle a_1, a_3\rangle S = \langle b_1, b_3\rangle S \wedge \langle a_1, a_3\rangle S \neq \emptyset) \vee \langle a_1, a_3\rangle = \langle b_1, b_3\rangle\}$
(38b) $e_{2cS,n} = \{\langle\langle a_1, a_3\rangle, \langle b_1, b_3\rangle\rangle : (|\langle a_1, a_3\rangle S| = |\langle b_1, b_3\rangle S| \wedge \langle a_1, a_3\rangle S \neq \emptyset) \vee \langle a_1, a_3\rangle = \langle b_1, b_3\rangle\}$

These modified definitions can be used to define the semantics of ditransitive sentences with two *the same CN* as in (36), that is sentences of the form (10d), repeated here:

(10d) $NP\ DTVP\ THE\ SAME\ CN_1\ Prep\ THE\ SAME\ CN_2$

The $DTVP$ denotes the ternary relation S, CN_1 denotes the set X and CN_2 denotes the set Y. The function $SAME(X, S)$-$SAME(Y, S)$, corresponding to the application of $SAME(Y, R)$ to the result of application of $SAME(X, R)$ to S, is defined in D16:

D16: $SAME(X, S)$-$SAME(Y, S)$=
(i)=$\{Q : Q \in PLR \wedge \neg 2(E) \subseteq Q\}$ if for any $R \in \Pi_{S_{X,Y}}(E)$ any $B \in \Pi^{ei}_R(E)$ is singular
(ii)=$\{Q : Q \in PLR \wedge \exists_R(R \in \Pi_{S_{X,Y}}(E)) \wedge \exists_B(B \in \Pi^{ei}_R(E) \wedge B \in PL \wedge Q(B) = 1\} \cup$
$\{Q : Q \in PLR \wedge \exists_{C \subseteq E} \forall_R(R \in \Pi^{ei}_{S_{X,Y}} \forall_B(B \in \Pi^{ei}_R(C \not\subseteq B \wedge \neg ALL(C) \subseteq Q\}$

Applying definition D16 to the relation S given in (37) we obtain two non-singular blocks $\{a_1, a_2\}$ and $\{a_1, a_2, a_3\}$ of which two different type $\langle 1\rangle$ quantifiers can be true.

The generalised determiner *the same* can also be used in *conjunctive sentences*, that is sentences in which a conjunction of many transitive verbs occurs. As Carlson (1987) points out, the sentence internal reading depends on distributing over separate events and a comparison of them. In sentences discussed until now a comparison of events due to actions performed by at least two agents is involved. Multiple events can be also obtained in the case of conjunctions of VPs. In such cases one agent performs actions giving rise to multiple events which are compared or differentiated.

As in the case of non-conjunctive sentences, two types of determiners can be used in conjunctive sentences: those expressing, roughly, the sameness of sets as in (39) and those expressing the sameness of cardinality of sets, as in (40):

(39) Leo bought and read the same books.

(40) Leo bought and read the same number of books.

In these sentences *the same* denotes a function of type $\langle 1, 2^2 : \langle 1 \rangle\rangle$. Strictly speaking, however, the number of relational arguments is not limited and thus functions denoted by *the same* in sentences expressing multiple events are of the type $\langle 1, 2^k : \langle 1 \rangle\rangle$, for any finite k. I will consider only the case of two relational arguments, that is when $k = 2$.

A possible approach to *the same* in conjunctive sentences, the approach taken for instance in Zuber (2011), is to use the notion of nominal case extension. Then sentence (39) would be true iff the set of books that Leo bought equals the set of books that Lea read. And the set of books that Leo read is obtained by applying the nominal extension of the quantifier denoted by *Leo* to the relation $READ$ and taking the intersection with the set of books. This proposal does not work, however, for all types of subject NPs. Consider for instance (41):

(41) Only Leo read and sold the same books.

It may happen that books that only Leo read are the same as books that only Leo sold but (41) may be false. This is the case for instance when $R = \{\langle l, b_1\rangle, \langle s_1, b_2\rangle\}$ and $S = \{\langle l, b_1\rangle, \langle s_1, b_2\rangle, \langle s_2, b_3\rangle\}$. One can see that the book that only l read is the same that the book that only l sold but (41) is not true because s_1 also read and sold the same book.

More correct and, interestingly, simpler, approach is to consider that the function involved in the interpretation of (39) or (40) has a set (of individuals) as output. This set is easy to specify. In other words we have the following type $\langle 1, 2^2 : 1\rangle$ functions involved in the interpretation of conjunctive sentences: in D17 we have the function interpreting (39) and in D18 the function interpreting (40):

D17: $SAME2(X, R, S) = \{x : xR \cap X = xS \cap X\}$
D18: $SAME2\text{-}N(X, R, S) = \{x : |xR \cap X| = |xS \cap X|\}$

It follows from D17 that (41) is false in the model indicated above. The reason is that in this model $SAME2(X, R, S) = \{l, s_1\}$ and thus it is not true that only l read (R) and bought (S) the same book.

Clearly sentences similar to (39) or (40) in which a plural subject NP occurs are ambiguous since they can be taken as conjunctive sentences or non-conjunctive ones in which an ordinary conjunction of two transitive verbs occurs. Thus (42) can mean either (43a) -conjunctive or (43b) non-conjunctive:

(42) Leo and Lea bought and read the same book.
(43a) Leo bought and read the same book and Lea bought and read the same book.
(43b) Leo bought and read exactly the same book as the book that was bought and read by Lea.

The tools presented above easily allow the disambiguisation of such sentences.

Though there are subtle differences between *(a) different* and *the same* which mean that they are not completely analogous (cf. Hardt and Mikkelsen 2015), these two generalised determiners are logically related. As the following examples show, *a different* can occur in transitive and ditransitive constructions which are quite similar to those in which *the same* occurs. In particular *(a) different* can express "set difference", "cardinality difference", multiple events set difference and multiple events cardinality difference:

(44) Leo and Lea solved different problems
(45) Most students solved a different number of problems.
(46) Leo invented and solved different problems.
(47) Leo invented and solved a different number of problems.

One can distinguish at least two readings of *different* according to whether the corresponding sets are just different or whether they are "strongly different" that is their intersection is empty. For instance if the sets of problems that Leo and Lea solved are not the same then (44) is true on the weak reading of *different* and if these sets are disjoint then (44) is true on the strong reading of *different*.

The strong difference heavily depends on the argument X. For instance for (50) to be true there must be at least two problems. The weak reading of *(a) different* is related to the reading of *the same* via Boolean complementation:

D19: (i) $DIFF(X,R) = (SAME(X,R))'$, if $X \neq \emptyset$
(ii) $DIFF(X,R) = \{Q : Q \in PLR \wedge \neg 2(E) \subseteq Q\}$ if $X = \emptyset$
D20: (i) $DIFF\text{-}N(X,R) = (SAME\text{-}N(X,R))'$, if $X \neq \emptyset$
(ii) $DIFF\text{-}N(X,R) = \{Q : \neg 2(E) \subseteq Q\}$ if $X = \emptyset$
D21: (i) $DIFF2(X,R,S) = (SAME2(X,R,S))'$, if $X \neq \emptyset$
(ii) $DIFF2(X,R,S) = \emptyset$ if $X = \emptyset$
D22: (i) $DIFF2\text{-}N(X,R,S) = (SAME2\text{-}N(X,R,S))'$, if $X \neq \emptyset$
(ii) $DIFF2\text{-}N(X,R,S) = \emptyset$ if $X = \emptyset$

The above definitions correspond to examples (44) - (47) respectively.

I conclude this section by indicating that the determiners *the same* can take many CNs as argument. For instance (48) naturally means (49):

(48) Leo and Lea read the same novels and plays.
(49) Leo and Lea read the same novels and they read the same plays

In (48) *the same* denotes a type $\langle 1^2, 2 : \langle 1 \rangle \rangle$ function. It is a meet of two type $\langle 1, 2 : \langle 1 \rangle \rangle$ functions. In the following example a type $\langle 1^2, 2^2 : \langle 1 \rangle \rangle$ function is involved:

(50) Leo bought and read the same novels and plays.

The function interpreting (42) is also a meet of two type $\langle 1, 2^2 : \langle 1 \rangle \rangle$ functions.

I mention the above examples to show informally the parallelism between "ordinary" determiners and higher order determiners. Ordinary n-ary determiners have been extensively studied (Keenan and Moss 1986, Beghelli 1994). We know for instance that there are binary determiners which denote functions which are not Boolean compositions of functions denoted by unary quantifiers. It is not clear whether there such irreducible higher order determiners. A possible candidate is given in (51). Observe that (51) entails neither (52a) nor (52b):

(51) Leo and Lea read more of the same plays than novels.
(52a) Leo and Lea read the same plays.
(52b) Leo and Lea read the same novels.

As far as I can tell the proposal made here to use the notion of set partition cannot be applied to the analysis of (51).

In the next section similarities between simple and higher order determiners are indicated more formally.

4 Some properties

Functions discussed in the previous section are not, strictly speaking quantifiers because their output is not a truth-value. Intuitively, however, they involve quantification in some way and so it might be interesting to try to grasp their similarities with quantifiers. I will do this in two ways: by showing that they share many formal properties with quantifiers and by showing some inference patterns they share with quantifiers.

Consider first the function $WSAME(X, R)$ defined in (53). This function has been proposed by Zuber (2011) as yielding the semantics of *the same CN* in general:

(53) $WSAME(X, R) = \{Q : Q \in PL \wedge Q^d R \cap X \subseteq QR \cap X\}$

The following proposition indicates the relationship between $WSAME$ and $SAME$:

Proposition 4: $Ft(A) \in SAME(X, R)$ iff $Ft(A) \in WSAME(X, R)$ (for $|A| \geq 2$)
.
Proof: From left to right: Suppose *a contrario* that (i) $Ft(A) \in SAME(X, R)$ and (ii) $Ft(A) \notin WSAME(X, R)$. It follows from D13 and (i) that there is a block $B \in \Pi_{R_X}(E)$ such that $A \subseteq B$. It follows from (ii) that for some $c_1, c_2 \in A$ we have $c_1 R_X \neq c_2 R_X$. Contradiction.
From right to left: If $Ft(A) \in WSAME(X, R)$ then $Ft(A)^d R \cap X \subseteq Ft(A) R \cap X$. But then $\forall_{y,z \in A}(yR_X = zR_X)$. This means that for some block $B \in \Pi_{R_X}(E)$ we have $A \subseteq B$ and thus $Ft(A) \in SAME(X, R)$.

One can check that $Ft(A)$ in proposition 4 cannot be replaced by many other type $\langle 1 \rangle$ quantifiers. This means that the function $WSAME$ accounts for the semantics of sentences with *the same* only if their subject NP denotes a principal filter.

Definitions D13 and D14 allow us also to prove:

Proposition 5: If $X \neq \emptyset$ and $R \neq \emptyset$ then $Q \in SAME(X, R)$ or $\neg Q \in SAME(X, R)$ and $Q \in SAME\text{-}N(X, R)$ or $\neg Q \in SAME\text{-}N(X, R)$, for any $Q \in PLR$.

Other formal properties of functions denoted by constructions with *the same* are those which are related to conservativity and its sub-properties, generalised in a natural way. They are defined in section 2 above. Thus, using Proposition 3 it is easy to prove:

Proposition 6: Functions $SAME$, $SAME^{ei}$ and $DIFF$ are intersective.

Concerning the type $\langle 1, 2 : \langle 1 \rangle \rangle$ functions denoted by the determiner *the same number of* and *different number of* we have:

Proposition 7: The functions $SAME\text{-}N$ and $DIFF\text{-}N$ are cardinal.

Proof: We prove only that $SAME\text{-}N$ is cardinal. Given D8 we have to show that $SAME\text{-}N(X_1, R) = SAME\text{-}N(X_2, S)$ if (i): $\forall_{y \in E}(|X_1 \cap yR| = |X_2 \cap yS|)$ holds. But if (i) holds then $\Pi_{R_{X_1},n}(Y) = \Pi_{S_{X_2},n}(Y)$, for any $Y \neq \emptyset$

For type $\langle 1, 2^2 : 1\rangle$ involved in the interpretation of conjunctive sentences we have:

Proposition 8 (i): $SAME2(X, R, S)$ is intersective.
(ii) $SAME2\text{-}N(X, R, S)$ is cardinal.

Since cardinality and intersectivity imply conservativity, all the above functions are conservative.

Let us see some properties which can be used to form specific inference patterns. First, given the fact that $aR = aS$ iff $aR' = aS'$ and $|aR| = |aS|$ iff $|aR'| = |aS'|$ one can see that functions $SAME$, $SAME\text{-}N$, $SAME2$ and $SAME2\text{-}N$ behave the same way on their relational arguments as they do on the complements of these arguments. More precisely we have:

Proposition 9 (i): $SAME(X, R) = SAME(X, R')$.
(ii) $SAME\text{-}N(X, R)$=$SAME\text{-}N(X, R')$
(iii) $SAME2(X, R) = SAME2(X, R')$
(iv) $SAME2\text{-}N(X, R) = SAME2\text{-}N(X, R')$

A similar proposition does not hold for the the function $SAME^{ei}$.

The identity of truth values between (54a) and (54b) illustrates clause (i) of the Proposition 9 and the identity of truth values between (55a) and (55b) illustrates clause (ii):

(54a) The books that most students read are the same.
(54b) The books that most students did not read are the same.
(55a) The number of books that most students read is the same.
(55b) The number of books that most students did not read is the same.

Let us see now some properties which give rise to inferential patterns in which some of the functions discussed above are involved. Using Proposition 1 and definition D13 one can prove Proposition 10. It is illustrated by the inference from (56a) to (56b):

Proposition 10: Let $X_1 \subseteq X_2$. Then $MON{\cap}SAME(X_2, R) \subseteq MON{\cap}SAME(X_1, R)$

(56a) Some students/mosts students read the same novels.
(56b) Some students/mosts students read the same Japanese novels.

The next proposition concerns the relationship between $SAME$ and $DIFF$ restricted to principal filters. The pattern to which it gives rise is illustrated by the equivalence between (57a) and (57b) (cf. Keenan 2005):

Proposition 11: $Ft(A) \in SAME(X, R)$ iff $Ft(A)\neg \in DIFF(X, R)$

(57a) Every student read the same novel.
(57b) No students read a different novel.

The last pattern I want to indicate concerns sentences whose direct object NPs are formed by determiners denoting atomic intersective or co-intersective type $\langle 1,1\rangle$ quantifiers. Here are some examples: (58) entails (59).

(58) Leo and Lea/most students read every book/no book except *Exciting Humour*
(59) Leo and Lea/most students read the same book(s)

The object *every book* is formed by application of the determiner *every* to the CN *book*. The determiner *every* denotes the atom $EVERY$ of the algebra of co-intersective type $\langle 1,1\rangle$ quantifiers such that $EVERY(X)(Y)=1$ iff $X\cap Y'=\emptyset$. Similarly, the determiner *no... except Exciting Humor* denotes the atom NO_{EH} of intersective type $\langle 1,1\rangle$ quantifiers such that $NO_{EH}(X)(Y)=1$ iff $X\cap Y=\{EH\}$.

The following proposition justifies the pattern underlying the above inference:

Proposition 12: Let Q be monotone increasing, $Q\in PLR$, R be a binary relation and D_A be an atom of INT or of $CO\text{-}INT$. Then (i) entails (ii):
(i) $Q(D_A(X)_{acc}R)=1$
(ii) $Q\in SAME(X,R))$

Proof: Suppose that $C=D_A(X)_{acc}(R)$. Then, given D2, we have $C=\{y: X\cap yR = A\}$. If $Q(C)=1$ then $|C|\geq 2$. It is easy to see that there exists a block B of $\Pi_{R_X}(E)$ such that $C\subseteq B$. Since $Q(C)=1$ and because $Q\in MON$, we have $Q(B)=1$ and thus $Q\in SAME(X,R)$. If D_A is co-intersective we use Proposition 9 and reason in the similar way.

The entailments indicated in proposition 12 hold under the non-cumulative and subject wide-scope reading of the NP corresponding to the quantifier Q.

Similar inference patterns concerning "numerical sameness" can be obtained with the help of type $\langle 1,1\rangle$ cardinal and co-cardinal atomic quantifiers. The following proposition has a proof similar to Proposition 12:

Proposition 13: Let $Q\in MON$, $Q\in PLR$ and $EXACT_n$ be an atom of $CARD$ or of $CO\text{-}CARD$. Then (i) entails (ii):
(i) $Q(EXACT_n(X)_{acc}R)$
(ii) $Q\in SAME\text{-}N(X,R)$

Proposition 13 can be illustrated by the following sentences: (60) entails (61):

(60) Leo and Lea hug all but five teachers.
(61) Leo and Lea hug the same number of teachers.

Interestingly, similar inferences are not in general possible with conjunctive sentences. Thus (62) does not entail (63):

(62) Leo bought and read every book except *Exciting Humour.*
(63) Leo bought and read the same books.

By contraposing various implications underlying the above patterns we obtain a similar series of inference patterns concerning the determiner *different* and the functions $DIFF$ and $DIFF\text{-}N$. Easy examples of instances of such patterns are omitted.

Though the inferences presented above are not very deep, they can be considered as an additional justification for the analysis proposed here.

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